1D Harmonic oscillator

Two ways of solving this problem with the Hamiltonian given by

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \qquad k = m\omega^2 \qquad [1]$$

Recall H is an operator

(i) In schrödinger eq. $p = \frac{\hbar}{i}\frac{d}{dx}$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2\right)\varphi_n(x) = E_n \varphi_n(x) \qquad [2]$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \qquad \text{quadratic dep. on } x \qquad [3]$$

$$\varphi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}}\, e^{-\xi^2/2}\, H_n(\xi) \qquad [4]$$

Hermite polynomials

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

(ii) operator method

$\longrightarrow$ purely algebraic manipulation

no need to use $p = \frac{\hbar}{i}\frac{d}{dx}$

But to use $[x, p] = i\hbar$

$$[A, B] = AB - BA \qquad \text{commutator} \qquad [5]$$

Write $H = \frac{1}{2m}\left[p^2 + (m\omega x)^2\right]$

Define
$$a_\pm \equiv \frac{1}{\sqrt{2m\hbar\omega}}\left(\mp ip \pm m\omega x\right) \qquad [6]$$

$$[a_-, a_+] = 1 \qquad [7]$$

Can prove:
$$a_- a_+ = \frac{1}{\hbar\omega} H + \frac{1}{2} \qquad [8]$$

$$H = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)$$

From HW1: prob #3
$$a_+ |n\rangle = \sqrt{n+1}\, |n+1\rangle \qquad \text{raising operator}$$
$$a_- |n\rangle = \sqrt{n}\, |n-1\rangle \qquad \text{lowering operator}$$

define ground state $\quad a_- |0\rangle = 0$

Free particle in 1 D

$$H = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} \qquad V(x) \equiv 0 \text{ everywhere} \qquad [9]$$

$$H\varphi_E = E\varphi_E \qquad -\infty < x \le \infty$$

For the unbound problem, E is continuous

$$\varphi_E(x) = e^{ikx} \text{ or } e^{-ikx}$$

$$\varphi_E(x,t) = e^{ikx - iEt/\hbar} = e^{i(kx-\omega t)} \quad \text{wave} \rightarrow \qquad E = \hbar\omega \qquad [10]$$

$$\text{or } e^{-ikx - iEt/\hbar} = e^{i(kx+\omega t)} \leftarrow \text{wave}$$

The solutions are so-called plane waves. They are travelling waves.

Since $|\varphi_E|^2 = 1$ everywhere. It is not a good representation of a localized particle in motion.

Construct "wave packet"

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \phi(k)\, e^{i(kx-\omega t)} \qquad [11] \qquad \underbrace{\omega = \frac{E}{\hbar} = \frac{k^2}{2m}}$$

dispersive medium

In optics, you see similar form where $\omega = vk$

where $v = c$ in vacuum, the system is not dispersion.

If $\omega = \omega(k)$ dispersive

You get phase velocity $v_p = \frac{\omega}{k}$

If ω changes slowly vs k,

you can get group velocity $v_g = \frac{d\omega}{dk}$

F.T.
$$\begin{cases} \psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ikx}\, dk & [12] \\ \phi(k) = \frac{1}{\sqrt{2\pi}} \int \psi(x,0)\, e^{-ikx}\, dx & [13] \end{cases}$$

You can obtain $\sigma_x \sigma_k \geq 2\pi$ ← the equivalent of Heisenberg uncertainty Relation

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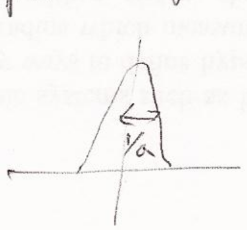
Example: A Gaussian wave packet (Example on p. 67)

$$\psi(x,0) = A\, e^{-ax^2} \qquad [14]$$



To find $\psi(x,t)$

Find $\phi(k)$ from [13]

plug in
$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx - \frac{E}{\hbar}t)}\, dk \qquad [15]$$

$$E = \frac{\hbar^2 k^2}{2m}$$

Integrate, get
$$\psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/[1+2i\hbar at/m]}}{\sqrt{1+2i\hbar at/m}} \qquad [16]$$

Find $|\psi(x,t)|^2$ get the width.

~~width~~ $w = \sqrt{\dfrac{a}{1+\left(\dfrac{2\hbar a t}{m}\right)^2}}$

~~The width grows with time. At large t~~

~~$w \simeq$~~

at $t=0$ $\quad \psi(x,0) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}$

$|\psi(x,0)|^2 = \left(\frac{2a}{\pi}\right)^{1/4} e^{2ax^2}$

large a, small width

$e^{-\left(\frac{x}{\beta}\right)^2}$ $\quad \beta = \frac{1}{2a}$

Time evolution of $|\psi(x,t)|^2$

Estimate

$$|\psi(x,t)|^2 \approx e^{\frac{-ax^2}{1+iy}} \cdot e^{\frac{-ax^2}{1-iy}} \qquad y = \frac{2\hbar a t}{m}$$

$$= e^{-ax^2\left(\frac{1}{1+iy} + \frac{1}{1-iy}\right)}$$

$$= e^{-\frac{2ax^2}{1+y^2}} = e^{-\left(\frac{x}{\beta\sqrt{1+y^2}}\right)^2} \qquad [17]$$

$$1+y^2 = 1 + \frac{2\hbar^2 a^2 t^2}{m^2}$$

For large time t, $\quad 1+y^2 \sim \frac{2\hbar^2 a^2 t^2}{m^2}$

The width is approximately given by

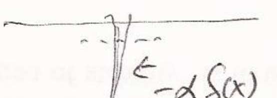
$$\beta'(t) = \beta \cdot \left(\frac{2\hbar^2 a^2}{m^2}\right)^{1/2} t$$

Conclusion: $\Bigg[$ The width grows with t

large a, starts with small width, spread faster

2.5 delta-function potential

Let $V(x) = -\alpha\,\delta(x)$ $\quad \alpha > 0$

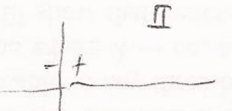


$$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} - \alpha\,\delta(x)\,\varphi = E\varphi \quad [18] \qquad E = -\frac{\hbar^2\beta^2}{2m}$$

Discontinuity of potential at $x=0$

Note $\varphi(x)$ should be finite. E is finite for

Discontinuity of $d\varphi/dx$ at $x=0$



solve region I : For bound state

$x<0$ $\quad \psi_I = e^{+\beta x}$ $\quad (x<0$

$x>0$ $\quad \psi_{II} = C e^{-\beta x}$

Integrate eq [18] $\int_{-\epsilon}^{\epsilon}$ on both sides of [18]

$$-\frac{\hbar^2}{2m}\left[\frac{d\varphi}{dx}\Big|_{\epsilon} - \frac{d\varphi}{dx}\Big|_{-\epsilon}\right] - \alpha\,\varphi(0) = E\int_{-\epsilon}^{\epsilon}\varphi(x) = 0$$

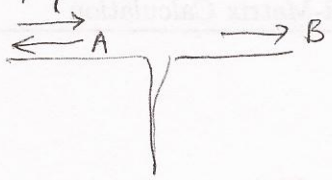
$$-\frac{\hbar^2}{2m}\left(\frac{d\varphi}{dx}\Big|_{+} - \frac{d\varphi}{dx}\Big|_{-}\right) = \alpha\,\varphi(0) = \alpha \qquad [19]$$

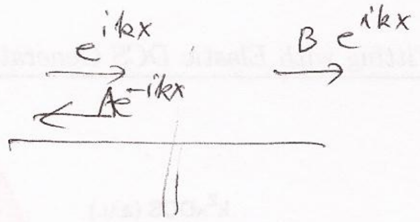
$$\psi_I(0) = \varphi_{II}(0) = C = 1$$

$$-\frac{\hbar^2}{2m}\left[-\beta - \beta\right] = \alpha \qquad \beta = \frac{\alpha m}{\hbar^2}$$

$$E = -\frac{\hbar^2}{2m}\left(\frac{\alpha m}{\hbar^2}\right)^2 = -\frac{m\alpha^2}{2\hbar^2} \qquad [20]$$

Scattering problem





$$E = \frac{\hbar^2 k^2}{2m}$$

$$\begin{cases} \psi_I = e^{ikx} + A e^{-ikx} \\ \psi_{II} = B e^{ikx} \end{cases} \qquad [21]$$

$\psi_I = \psi_{II}$ at $x = 0$

$$1 + A = B$$

$$\cancel{\text{Eq. [19]} \; (ikB - (ik - ikA)) =}$$

Eq [19] $$-\frac{\hbar^2}{2m}\left[ikB - ik(1-A)\right] = \alpha B$$

$$-\frac{\hbar^2}{2m} \cdot ik \left[B - 1 + A\right] = \alpha B$$

define $\beta = \frac{m\alpha}{\hbar^2 k}$ $\qquad -\frac{i}{\beta} = \frac{B}{B-1}$

$$B = \frac{-i}{-i-\beta} = \frac{1}{1-i\beta}$$

$$\begin{cases} |B|^2 = \frac{1}{1+\beta^2} & \text{transmison} \\ |A|^2 = \frac{\beta^2}{1+\beta^2} & \text{reflection} \end{cases} \qquad [22]$$

$$|A|^2 + |B|^2 = \underline{1}$$

Note: different results from Q.M. as compared to classical

In [2II] the particle can be reflected back

At low energy $k \to 0$ $\beta \to \infty$ mostly reflects back

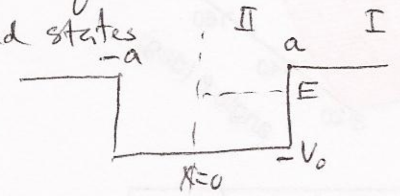
Also, if we change $\alpha \to -\alpha$, the same reflection/transmission coefficients!!

Replaced

See next file

2.6. Finite square well

(1) bound states



$\begin{cases} \text{even} \\ \text{odd} \end{cases}$ solutions $\boxed{\text{Parity}}$

I: $\psi_I = e^{-\beta x}$ $\qquad \dfrac{\hbar^2 \beta^2}{2m} = -E$

II: $\psi_{II} = A \cos l x$ $\qquad \dfrac{\hbar^2 l^2}{2m} = E + V_0$

continuity at $x = a$

two unknowns A, E

(2) scattering problem

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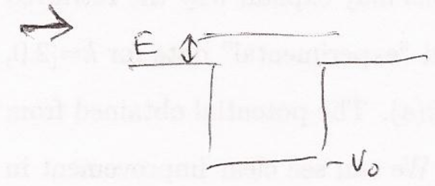
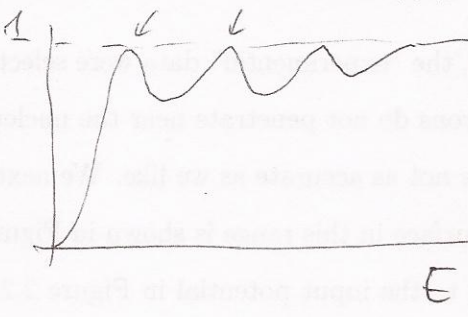


Resonance



Eq. (2.170)

$$\frac{2a}{\hbar}\sqrt{2m(E+V_0)} = n\pi$$ Resonance condition

(3) potential step (homework)