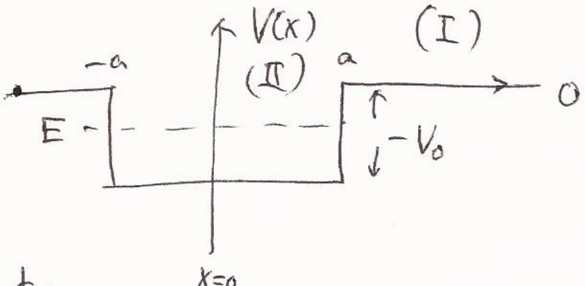
The finite square well



Bound states:

Since $V(x) = V(-x)$ we look for solution in regin I and II $(x \geq 0)$ only

Regin I: $$-\frac{\hbar^2}{2m}\frac{d^2\psi_I}{dx^2} = E\psi_I = -\frac{\hbar^2\beta^2}{2m}\psi_I \qquad \boxed{E = -\frac{\hbar^2\beta^2}{2m}}$$

Reg. II. $$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} - V_0\psi_{II} = E\psi_{II}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} = (E+V_0)\psi_{II} = \frac{\hbar^2 l^2}{2m}\psi_{II} \qquad l^2 = \frac{2m(E+V_0)}{\hbar^2}$$

Boundary conditions:

$x \to \infty$ $\quad \psi_I = e^{-\beta x}$

$x = a$ $\quad \sin lx$ and $\cos lx$

Even solution: $\psi_{II} = B\cos lx$

at $x = a$

Continuity: $B\cos la = e^{-\beta a}$

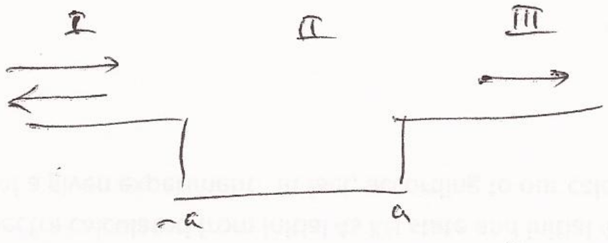
derivative $-lB\sin la = -\beta e^{-\beta a}$

$$\boxed{l\tan la = \beta}$$ ← solve this eq.

Odd solution: Similar method

Scattering states $(E > 0)$

Cannot use symmetry directly, since the symmetry is broken by boundary condition.



$$\psi_I = e^{ikx} + A e^{-ikx} \qquad E = \frac{\hbar^2 k^2}{2m}$$

$$\psi_{II} = B \sin lx + C \cos lx$$

$$\psi_{III} = D e^{ikx}$$

Reflection coeff. $R = |A|^2$

Transmission coeff. $T = |D|^2$

Four unknowns, A, B, C, D. Four equations from boundary conditions at $x = a$ and $x = -a$

Final results, [Eq. 2.169]

$$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)$$

$T = 1$ when

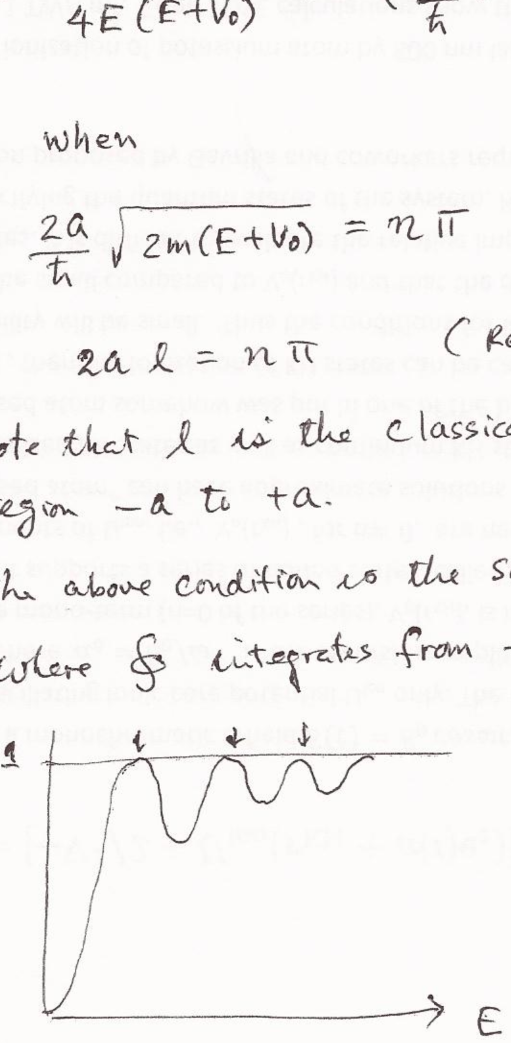
$$\left.\frac{2a}{\hbar}\sqrt{2m(E+V_0)} = n\pi \right\} \text{ resonance condition}$$

$2al = n\pi$ (Recall the definition)

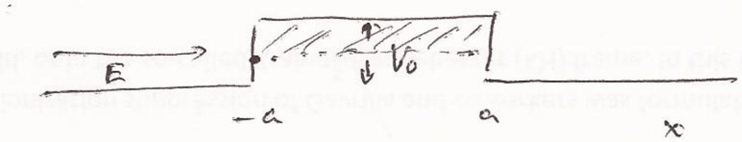
Note that l is the classical momentum in the region $-a$ to $+a$.

The above condition is the same $\oint p\,dx = 2n\pi$ — classical quantization

where $\oint$ integrates from $-a$ to a and back



Finite potential step



Homework or prob. 2.33 (p. 83)

For $E < V_0$

$$T^{-1} = 1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right)$$

When $E << V_0$ $\quad \sinh x \sim \frac{1}{2} e^x$ for $x >> 1$

$$T^{-1} = \frac{V_0}{4E}\left(\frac{1}{4}\right) e^{2x} \qquad x = \frac{2a}{\hbar}\sqrt{2m(V_0-E)}$$

$$T = \frac{16E}{V_0} e^{-2x}$$

↑ Transmission is an exponential function

$\boxed{\text{Tunneling}}$ ← α-decay, Ionization in E-field

A quantum particle can tunnel out of the barrier.

The rate ~ related to the shaded area in the figure above.

More can be learned from working out Prob. 2.33 for $E > V_0$