The finite square well

\[ V(x) = \begin{cases} 
-\infty & \text{if } x < -a \\
0 & \text{if } -a \leq x \leq a \\
\infty & \text{if } x > a 
\end{cases} \]

Bound states:

Since \( V(x) = V(-x) \) we look for solutions in regions I and II \((x > 0)\) only.

\[ \text{Reg. I: } -\frac{\hbar^2}{2m} \frac{d^2 \psi_I}{dx^2} = E \psi_I = -\frac{\hbar^2 \beta^2}{2m} \psi_I \]

\[ E = -\frac{\hbar^2 \beta^2}{2m} \]

\[ \text{Reg. II: } -\frac{\hbar^2}{2m} \frac{d^2 \psi_II}{dx^2} - V_0 \psi_II = E \psi_II \]

\[ -\frac{\hbar^2}{2m} \frac{d^2 \psi_II}{dx^2} = (E + V_0) \psi_II = \frac{\hbar^2 \ell^2}{2m} \psi_II \]

\[ \ell = \frac{2m(E + V_0)}{\hbar^2} \]

Boundary conditions:

\[ x \to \infty \quad \psi_I = e^{-\beta x} \]

\[ x = a \quad \text{sin}x \text{ and cos}x \]

Even solution: \( \psi_II = B \cos x \)

at \( x = a \) continuity:

\( B \cos a = e^{-\beta a} \)

derivative:

\( -\beta B \sin a = -\beta e^{-\beta a} \)

\[ \ell \tan a = \beta \]

Odd solution: similar method

Scattering states \((E > 0)\)

Cannot use symmetry directly, since the symmetry is broken by boundary condition.
\[ E = \frac{\hbar^2 k^2}{2m} \]

\[ V_{\text{in}} = B \sin kx + C \cos kx \]
\[ V_{\text{out}} = D e^{-ikx} \]

Reflection coeff. \( R = |A|^2 \)
Transmission coeff. \( T = |D|^2 \)

Four unknowns, \( A, B, C, D \). Four equations from boundary conditions at \( x = a \) and \( x = -a \).

Final results, \( \text{[Eq. 2.169]} \)

\[ T = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2 \left( \frac{2a \sqrt{2m(E+V_0)}}{\hbar} \right) \]

\[ T = 1 \quad \text{when} \quad \frac{2a \sqrt{2m(E+V_0)}}{\hbar} = n \pi \]

\[ 2a \ell = n \pi \quad \text{(Recall the definition)} \]

Note that \( \ell \) is the classical momentum in the region \(-a \leq x \leq a\).

The above condition is the same as \( \int p \, dx = 2n \pi \)
where \( \int \) integrates from \(-a \) to \( a \) and back.

\[ T \quad \ell \quad E \]
Finite potential step

\[
E \rightarrow \left\{ \begin{array}{ll} V_0 & 0 < x < a \\ -V_0 & a < x < -a \end{array} \right.
\]

Homework or prob. 2.33 (p. 83)

For \( E < V_0 \)

\[
T^{-1} = 1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 \left( \frac{2a}{\hbar} \sqrt{2m(V_0 - E)} \right)
\]

When \( E << V_0 \), \( \sinh x \approx \frac{e^x}{2} \) for \( x \gg 1 \)

\[
T^{-1} = \frac{V_0}{4E} \left( \frac{1}{4} \right) e^{2x} \quad x = \frac{2a}{\hbar} \sqrt{2m(V_0 - E)}
\]

\[
T = \frac{16E}{V_0} e^{-2x}
\]

\[\text{Transmission is an exponential function}\]

\[\text{Tunneling} \quad \leftarrow \quad \text{Ionization in E-field}\]

A quantum particle can tunnel out of the barrier.

The rate is related to the shaded area in the figure above.

More can be learned from working out prob. 2.33 for \( E > V_0 \)