Spin angular momentum (chapter 4)

Spin angular momentum $\vec{S}$.

Superpose $[S_x, S_y] = i\hbar S_z$, $[S_y, S_z] = i\hbar S_x$, $[S_z, S_x] = i\hbar S_y$ (1)

Define $S^2 = S_x^2 + S_y^2 + S_z^2$

$$[S^2, S_x] = [S_x^2 + S_y^2 + S_z^2, S_x]$$

$$= [S_y^2, S_x] + [S_z^2, S_x] \quad (2)$$

Note: $[AB, C] = A[B,C] + [A,C]B$

$$(2) = S_y[S_y, S_x] + [S_y, S_x]S_y + S_z[S_z, S_x] + [S_z, S_x]S_z$$

$$= S_y(-i\hbar)S_z - i\hbar S_z S_y + i\hbar S_z S_y + i\hbar S_y S_z$$

$$= 0$$

Thus $[S^2, S_x] = 0$

Similarly $[S^2, S_y] = 0$

$[S^2, S_z] = 0$

If two operators commute, they have common eigenfunctions

For example: $[S^2, S_z] = 0$

Let $S^2 f = \lambda f$

$$S^2(S_z f) = S_z(S^2 f) = \lambda S_z f$$

Thus $S_z f$ is also an eigenstate of S^2

possible only $S_z f = \mu f$

i.e., proportional to f.

Define $S_+ = S_x + i S_y$ similar to a_+, a_- before

$S_- = S_x - i S_y$

$[S^2, S_+] = 0 \qquad [S^2, S_-] = 0$

$[S_z, S_+] = [S_z, S_x] + i[S_z, S_y] = i\hbar S_y - i^2\hbar S_x$

$= \hbar S_+$

$[S_z, S_-] = -\hbar S_-$

Let $\begin{cases} S^2 f = \lambda f \\ S_z f = \mu f \end{cases}$

Since $[S^2, S_\pm] = 0$

$S^2(S_\pm f) = \lambda(S_\pm f)$

$[S_z, S_\pm] = \pm\hbar S_\pm$

$S_z S_\pm = S_\pm S_z \pm \hbar S_\pm$

$S_z S_\pm f = \mu S_\pm f \pm \hbar S_\pm f = (\mu \pm \hbar) S_\pm f$

Thus eigenvalue of S_z for $S_\pm f$ increase (decreases) by $\pm\hbar$

S_+ : raising operator

S_- : lowering operator

Since λ is finite, one cannot use S_+ indefinite times.

At some point $S_+ f_t = 0$

One can show

$$S^2 = S_\pm S_\mp + S_z^2 \mp \hbar S_z$$

$$S^2 f_t = (S_- S_+ + S_z^2 + \hbar S_z) f_t$$
$$= (0 + \hbar^2 l^2 + \hbar^2 l) f_t$$
$$= \hbar^2 l(l+1) f_t \qquad (3)$$

Thus $\lambda = l(l+1)\hbar^2$

The same procedure can be applied to S_-. There will be a bottom state where

$$S_- f_b = 0$$

Let $S_z f_b = \hbar \bar{l} f_b$

$$S^2 f_b = \lambda f_b$$

Use

$$S^2 f_b = (S_+ S_- + S_z^2 - \hbar S_z) f_b$$
$$= (0 + \hbar^2 \bar{l}^2 - \hbar^2 \bar{l}) f_b$$
$$= \hbar^2 \bar{l}(\bar{l} - 1) f_b \qquad (4)$$
$$= \lambda f_b$$

Compare (3) and (4)

$$\bar{l} = -l$$

Thus there are $2l+1$ solutions of S_z

Define these eigensolutions

$$\boxed{\begin{aligned} S^2 |sm\rangle &= \hbar^2 s(s+1) |sm\rangle \\ S_z |sm\rangle &= m\hbar |sm\rangle \end{aligned}}$$

where $m = -s, -s+1, \cdots 0, \cdots s$

since $2s+1$ be an integer, $s = \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$

Spin 1/2

The smallest $s = 1/2$. Many kinds of particles have spin $= 1/2$

Spin has no classical analog. It has nothing to do with the spinning of a particle. We consider spin has no volume or mass. It is for a point particle.

If $s = 1/2, 3/2, 5/2, \cdots$ they are called Fermi particles (or Fermi-Dirac), Fermions

If $s = 0, 1, 2, \cdots$ Bosons

Such distinctions are important for a system of identical particles

Spin 1/2 particles can be expressed as spinors.

Since there are only two $(2s+1 = 2\times\frac{1}{2}+1)$ eigenstates

$$S^2|sm\rangle = s(s+1)\hbar^2|sm\rangle \qquad s = 1/2,$$

$$S_z|sm\rangle = m\hbar|sm\rangle \qquad m = -1/2, 1/2$$

$$|sm\rangle = |m\rangle \Rightarrow |+\rangle = \begin{pmatrix}1\\0\end{pmatrix} = \chi_+$$

$$|-\rangle = \begin{pmatrix}0\\1\end{pmatrix} = \chi_-$$

Any states $\chi = \begin{pmatrix}a\\b\end{pmatrix} = a\chi_+ + b\chi_-$

where $|a|^2 + |b|^2 = 1$, both a and b are complex numbers

Our Hilbert space is two-dimensional

In the space spanned by (χ_+, χ_-) or $(|+\rangle, |-\rangle)$, every operator is represented by a matrix (2×2)

~~Clearly~~ clearly $S^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$\boxed{\text{Very useful eq.} \quad S_\pm |sm\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\;|s\; m\pm 1\rangle}$$

$$\begin{cases} a_+|n\rangle = \sqrt{n+1}\,|n+1\rangle \\ a_-|n\rangle = \end{cases}$$

$$S_+|\tfrac{1}{2} m\rangle = \hbar\sqrt{3/4 - m(m+1)}\;|\tfrac{1}{2}\; m+1\rangle \qquad S_-|\tfrac{1}{2} m\rangle = \sqrt{\tfrac{3}{4} - m(m-1)}\;|\tfrac{1}{2}\; m-1\rangle$$

$$S_+|+\rangle = 0 \qquad S_-|+\rangle = \hbar|-\rangle$$

$$S_+|-\rangle = \hbar|+\rangle \qquad S_-|-\rangle = 0$$

$$S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{Matrix representation}$$

$$S_x = \frac{1}{2}(S_+ + S_-) \qquad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Define Pauli matrices

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Recall in this representation, χ_+ and χ_- are eigenstates of S_z

$$S_z\chi_+ = \frac{\hbar}{2}\chi_+ \qquad S_z\chi_- = -\frac{\hbar}{2}\chi_-$$

Find eigenstates of S_x : $\quad S_x\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$

In matrix representation

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$|+\rangle_x = \frac{1}{\sqrt{2}}\chi_+ + \frac{1}{\sqrt{2}}\chi_- = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$|-\rangle_x = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

If you measure S_z for the $|+\rangle_x$ state, what is the expectation value ?

$$_x\langle + | S_z | + \rangle_x = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = 0$$