5.5 Addition of two angular momenta.

(a) Adding up two spin 1/2 particles

$\vec{S} = \vec{S}_1 + \vec{S}_2$ $\quad S_z = S_{1z} + S_{2z}$

Choose eigenstates of $\vec{S}_1^2, S_{1z}$: $\underset{\uparrow}{\chi_1^+}, \underset{\downarrow}{\chi_1^-}$ for $m = +\frac{1}{2}, -\frac{1}{2}$ respectively

Four possible combinations for the two spin 1/2 particles

1 2	α	m
$\uparrow\uparrow$	$\|++\rangle$	1
$\uparrow\downarrow$	$\|+-\rangle$	0
$\downarrow\uparrow$	$\|-+\rangle$	0
$\downarrow\downarrow$	$\|--\rangle$	-1

$S_z|\chi_1\chi_2\rangle = m\hbar|\chi_1\chi_2\rangle$

Define $S_- = S_{1-} + S_{2-}$

$S_-(\uparrow\uparrow) = (S_{1-} + S_{2-})|++\rangle$

$= (\hbar|-\rangle)|+\rangle + |+\rangle(\hbar|-\rangle)$

$= \hbar(|-+\rangle + |+-\rangle)$

For $m = 1$ or $m = -1$

$S^2 = S_1^2 + S_2^2 + 2\vec{S}_1\cdot\vec{S}_2$

$\vec{S}_1\cdot\vec{S}_2 = S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}$

$(\vec{S}_1\cdot\vec{S}_2)$ to find

$S_x|+\rangle = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$

$S_x|-\rangle =$

$S^2|++\rangle = 2\hbar^2|++\rangle$

Denote eigenstates of S^2 and S_z to be $|SM\rangle$

triplet
$$\begin{cases} |11\rangle = |++\rangle \\ |10\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) & S=1 \\ |1-1\rangle = |--\rangle \end{cases}$$

singlet $\quad |00\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \quad S=0$

We will not go into the derivation, but

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

Eigenstates of S^2 and S_z, $|s\,m\rangle$

$$s = |(s_1 - s_2)|, \ldots (s_1 + s_2)$$

The eigenstates can be expressed in term of $|s_1 m_1\rangle |s_2 m_2\rangle$

where $|s_1 m_1\rangle$ are eigenstates of S_1^2, S_{1z}

$|s_2 m_2\rangle$ $\quad S_2^2$, S_{2z}

$$\boxed{|s\,m\rangle = \sum_{m_1 + m_2 = m} \underbrace{C^{s_1 s_2 s}_{m_1 m_2 m}}_{\text{Clebsch-Gordan coefficients}} |s_1 m_1\rangle |s_2 m_2\rangle}$$

C-G coefficients

less than 1

real number

This equation can be used for any angular momenta.

The above equation can also be used ~~a~~

$$\boxed{|s_1 m_1\rangle |s_2 m_2\rangle = \sum_{s} C^{s_1 s_2 s}_{m_1 m_2 m} |s\,m\rangle}$$

note $m = m_1 + m_2$ from $S_z = S_{1z} + S_{2z}$

The C-G coefficients are tabulated in many books

but also available ~~comp~~ in web.

just search Clebsch-Gordan Coefficients