Electron in a magnetic field

The magnetic dipole moment $\vec{\mu}$ is proportional to its spin angular momentum $\vec{S}$

$$\vec{\mu} = \underset{\uparrow \text{ gyromagnetic ratio}}{\gamma} \vec{S}$$

A magnetic dipole in a magnetic field $\vec{B}$ has

$$H = -\vec{\mu} \cdot \vec{B}$$

Example 1: Larmor precession

Let $B = B_0 \hat{z}$

$$H = -\gamma \vec{S} \cdot \vec{B} = -\gamma B_0 S_z = -\gamma B_0 \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H \chi_\pm = \mp (\gamma B_0 \hbar/2) \chi_\pm = E_\pm \chi_\pm$$

Time evolution of a state

at $t=0$ $\quad \chi = a\chi_+ + b\chi_-$

$$\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar}$$

$$= \begin{pmatrix} a\, e^{i\gamma B_0 t/2} \\ b\, e^{-i\gamma B_0 t/2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\alpha}{2}\, e^{iAt} \\ \sin\frac{\alpha}{2}\, e^{-iAt} \end{pmatrix} \qquad A = \gamma B_0/2$$

Note $\chi(t)$ is not an observable

The spin angular momentum $\langle \vec{S} \rangle = \langle \chi | \vec{S} | \chi \rangle$

$$\langle S_x \rangle = \left(\cos\tfrac{\alpha}{2} e^{-iAt}, \sin\tfrac{\alpha}{2} e^{iAt}\right) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} e^{iAt} \\ \sin\frac{\alpha}{2} e^{-iAt} \end{pmatrix}$$

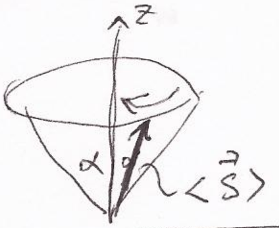
$$= \frac{\hbar}{2} \left(\cos\tfrac{\alpha}{2} e^{-iAt}, \sin\tfrac{\alpha}{2} e^{iAt}\right) \begin{pmatrix} \sin\frac{\alpha}{2} e^{-iAt} \\ \cos\frac{\alpha}{2} e^{iAt} \end{pmatrix}$$

$$= \frac{\hbar}{2} \left(\cos\tfrac{\alpha}{2} \sin\tfrac{\alpha}{2}\right)\left(e^{-2iAt} + e^{2iAt}\right) = \frac{\hbar}{2} \sin\alpha \cos(\gamma B_0 t)$$

$$\langle S_y \rangle = -\frac{\hbar}{2} \sin\alpha \sin(\gamma B_0 t)$$

$$\langle S_z \rangle = \frac{\hbar}{2} \cos\alpha$$

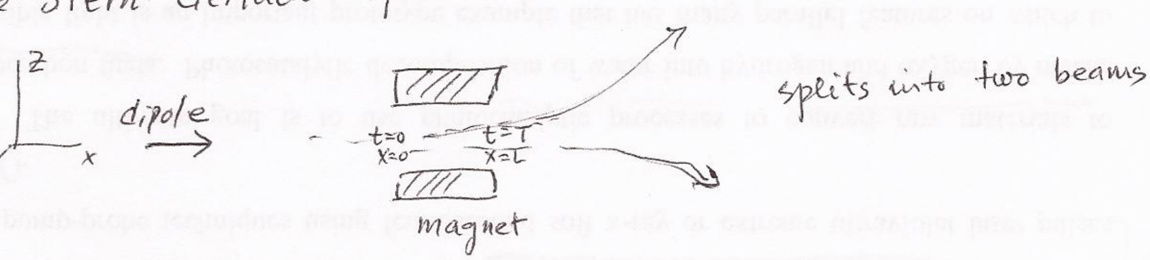
$$\langle \vec{S} \rangle = \frac{\hbar}{2}\left[\sin\alpha \cos(\gamma B_0 t)\,\vec{i} - \sin\alpha \sin(\gamma B_0 t)\,\vec{j} + \cos\alpha\,\vec{k}\right]$$



rotation ~ $\cos\omega t$ $\boxed{\omega = \gamma B_0}$ → Larmor frequency

$\langle \vec{S} \rangle$ precesses about $\hat{z} = \vec{k}$ as in classical physics

the Stern-Gerlach experiment



The magnets create a inhomogeneous magnetic field along z

$$F = -\nabla U = -\nabla(-\vec{\mu}\cdot\vec{B})$$

Assume $B_z = B_0 + \alpha z$

$$F = \gamma \alpha S_z$$

$$H = -\vec{\mu}\cdot\vec{B} = -\gamma(B_0 + \alpha z) S_z$$

At $t = 0$, $x = 0$ $\qquad E_\pm = \mp\gamma(B_0 + \alpha z)\frac{\hbar}{2}$

Write $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\chi(t) = \begin{pmatrix} a e^{i\gamma(B_0+\alpha z)t/2} \\ b e^{-i\gamma(B_0+\alpha z)t/2} \end{pmatrix}$$

upon exit the B-field, the particle gains a momentum P_z

$$P_z = \pm \gamma \alpha T \hbar/2$$

where $T = \frac{L}{v}$ is the time the particle spent in the gap.

The Stern-Gerlach experiment provides the direct proof the quantization of angular momenta and that total angular momentum of a particle can take half-integer values.

The following relations or results are easy to prove. You can confirm them but I will not collect them. All for $S = 1/2$.

(1) About Pauli matrices

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I \leftarrow \text{identity matrix}$$

$$\sigma_x \sigma_y = -\sigma_y \sigma_x \qquad \text{cyclic}$$

$$\sigma_x \sigma_y = i\, \sigma_z \qquad \text{cyclic}$$

(2) On page 110, there is the so-called Generalized Uncertainty Principle

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} [A, B] \right)^2 \qquad (1)$$

Let $A = S_x$ $\qquad [S_x, S_y] = i\hbar S_z$

$B = S_y$

Prove the relation (1) above for a general spin 1/2 state $|\psi\rangle = a\chi_+ + b\chi_- \stackrel{\text{or}}{=} \begin{pmatrix} a \\ b \end{pmatrix}$