HW 4

1. The energy levels of a 3D infinite spherical well with radius a are given in [4.50] of Griffiths

$$E_{nl} = \frac{\hbar^2}{2ma^2}\beta_{nl}^2$$

Where β_{nl} is the nth zero of the lth spherical Bessel function
You can use Mathematica to find β_{nl}.

$l=0$ $\beta_{10} = \overset{3.142}{\pi}$ (1) $\beta_{20} = \overset{6.283}{2\pi}$ (4) $\beta_{30} = \overset{9.425}{3\pi}$ (10) $\beta_{40} = \overset{12.566}{4\pi}$

$l=1$ $\beta_{11} = 4.493$ (2) $\beta_{21} = 7.725$ (6) $\beta_{31} = 10.904$ $\beta_{41} = 14.066$

$l=2$ $\beta_{12} = 5.763$ (3) $\beta_{22} = 9.095$ (8) $\beta_{32} = 12.323$ $\beta_{42} = 15.515$

$l=3$ $\beta_{13} = 6.988$ (5) $\beta_{23} = 10.417$ (11) $\beta_{33} = 13.698$ $\beta_{43} = 16.924$

$l=4$ $\beta_{14} = 8.183$ (7) $\beta_{24} = 11.705$ $\beta_{34} = 15.040$ $\beta_{44} = 18.301$

$l=5$ $\beta_{15} = 9.356$ (9) $\beta_{25} = 12.967$ $\beta_{35} = 16.355$ $\beta_{45} = 19.653$

$l=6$ $\beta_{16} = 10.513$ $\beta_{26} = 14.207$ $\beta_{36} = 17.648$

		Energy [in unit $\frac{\hbar^2}{2ma^2}$]	wavefunction	degeneracy $=2l+1$
1	$n=1,\ l=0$	9.87	$\psi_{100} = A_{10} j_0(\beta_{10} r/a) Y_0^0(\theta,\phi)$	1 (none)
2	$n=1,\ l=1$	20.19	$\psi_{11m} = A_{11} j_1(\beta_{11} r/a) Y_1^m(\theta,\phi)$	3
3	$n=1,\ l=2$	33.21	$\psi_{12m} = A_{12} j_2(\beta_{12} r/a) Y_2^m(\theta,\phi)$	5
4	$n=2,\ l=0$	39.48	$\psi_{200} = A_{20} j_0(\beta_{20} r/a) Y_0^0(\theta,\phi)$	1 (none)
5	$n=1,\ l=3$	48.83	$\psi_{13m} = A_{13} j_3(\beta_{13} r/a) Y_3^m(\theta,\phi)$	7
6	$n=2,\ l=1$	59.68	$\psi_{21m} = A_{21} j_1(\beta_{21} r/a) Y_1^m(\theta,\phi)$	3
7	$n=1,\ l=4$	66.96	$\psi_{14m} = A_{14} j_4(\beta_{14} r/a) Y_4^m(\theta,\phi)$	9
8	$n=2,\ l=2$	82.72	$\psi_{22m} = A_{22} j_2(\beta_{22} r/a) Y_2^m(\theta,\phi)$	5
9	$n=1,\ l=5$	87.53	$\psi_{15m} = A_{15} j_5(\beta_{15} r/a) Y_5^m(\theta,\phi)$	11
10	$n=3,\ l=0$	88.83	$\psi_{30m} = A_{30} j_0(\beta_{30} r/a) Y_0^0(\theta,\phi)$	1 (none)

2. The general solution inside the potential well is given in [4.45] of Griffiths

$$u(r) = A r j_l(kr) + B r n_l(kr)$$

Where j_l is the spherical Bessel function of order l and n_l is the spherical Neumann function of order l.

Now the potential well is between $a \leq r \leq b$. The boundary condition is that $u(r)$ should vanish at $r=a$ and at $r=b$.

$$\begin{cases} u(a) = A a j_l(ka) + B a n_l(ka) = 0 & (1) \\ u(b) = A b j_l(kb) + B b n_l(kb) = 0 & (2) \end{cases}$$

$$(2) \Rightarrow \quad B = -\frac{A j_l(kb)}{n_l(kb)}$$

Substitute this back to (1) one gets

$$j_l(ka)\, n_l(kb) - j_l(kb)\, n_l(ka) = 0$$

Therefore allowed k, thus E, should be such that the above equation is fullfilled.

3. (a) Radial wavefunction of the infinite spherical well

$$R(r) = \frac{u(r)}{r} = A_{nl}\, j_l\left(\frac{\beta_{nl}}{a} r\right)$$

normalization condition $\quad 1 = \int_0^a |R|^2 r^2 dr$

$\boxed{l=0}\qquad j_0(x) = \dfrac{\sin x}{x}$

$$1 = \int_0^a |A_{n0}|^2 \left|j_0\left(\frac{\beta_{n0}}{a} r\right)\right|^2 r^2 dr$$

set $\quad x \equiv \frac{\beta_{n0}}{a} r, \quad r = \frac{a}{\beta_{n0}} x, \quad dr = \frac{a}{\beta_{n0}} dx$

$$1 = |A_{n0}|^2 \int_0^{\beta_{n0}} |j_0(x)|^2 \left(\frac{a}{\beta_{n0}}\right)^3 x^2 dx$$

$$= \left(\frac{a}{\beta_{n0}}\right)^3 |A_{n0}|^2 \int_0^{\beta_{n0}} \frac{\sin^2 x}{\cancel{x^2}} \cancel{x^2}\, dx$$

$$= \left(\frac{a}{\beta_{n0}}\right)^3 |A_{n0}|^2 \left[\frac{x}{2} - \frac{1}{4}\sin(2x)\right]_0^{\beta_{n0}}$$

$$= \left(\frac{a}{\beta_{n0}}\right)^3 |A_{n0}|^2 \left[\frac{\beta_{n0}}{2} - \frac{1}{4}\sin(2\beta_{n0})\right]$$

$$= |A_{n0}|^2 \cdot \frac{a^3}{2\beta_{n0}^2}\left[1 - \frac{\sin(2\beta_{n0})}{2\beta_{n0}}\right]$$

$$\Rightarrow \quad A_{n0} = \frac{\sqrt{2}\,\beta_{n0}}{a^{3/2}\sqrt{1 - \sin(2\beta_{n0})/2\beta_{n0}}}$$

for $n=1$, $\beta_{10} = \pi$, $\quad A_{10} = \sqrt{2}\pi / a^{3/2}$

for $n=2$, $\beta_{20} = 2\pi$, $\quad A_{20} = 2\sqrt{2}\pi / a^{3/2}$

$$R_{10}(r) = A_{10}\, j_0\left(\frac{\beta_{10} r}{a}\right) = \frac{\sqrt{2}\pi}{a^{3/2}} \cdot \frac{\sin\left(\frac{\pi r}{a}\right)}{\frac{\pi r}{a}} = \sqrt{\frac{2}{a}}\, \frac{\sin\left(\frac{\pi r}{a}\right)}{r}$$

$$R_{20}(r) = A_{20}\, j_0\left(\frac{\beta_{20} r}{a}\right) = \frac{2\sqrt{2}\pi}{a^{3/2}} \cdot \frac{\sin\left(\frac{2\pi r}{a}\right)}{\frac{2\pi r}{a}} = \sqrt{\frac{2}{a}} \cdot \frac{\sin\left(\frac{2\pi r}{a}\right)}{r}$$

$\boxed{l=1}$ $\quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$

$$1 = \int_0^a |A_{n1}|^2 \left| j_1\left(\frac{\beta_{n1}}{a} r\right)\right|^2 r^2 dr$$

set $x \equiv \frac{\beta_{n1}}{a} r$, $r = \frac{a}{\beta_{n1}} x$, $dr = \frac{a}{\beta_{n1}} dx$

$$1 = |A_{n1}|^2 \int_0^{\beta_{n1}} |j_1(x)|^2 \cdot \left(\frac{a}{\beta_{n1}}\right)^3 x^2 dx$$

$$= |A_{n1}|^2 \cdot \left(\frac{a}{\beta_{n1}}\right)^3 \int_0^{\beta_{n1}} \left[\frac{\sin x}{x^2} - \frac{\cos x}{x}\right]^2 x^2 dx$$

$$= |A_{n1}|^2 \left(\frac{a}{\beta_{n1}}\right)^3 \int_0^{\beta_{n1}} \left(\frac{\sin x}{x} - \cos x\right)^2 dx$$

$$= |A_{n1}|^2 \cdot \left(\frac{a}{\beta_{n1}}\right)^3 \cdot \frac{\beta_{n1}\sin(2\beta_{n1}) + 2\cos(2\beta_{n1}) + 2\beta_{n1}^2 - 2}{4\beta_{n1}}$$

$$A_{n1} = \frac{2\beta_{n1}^2}{a^{3/2}\sqrt{\beta_{n1}\sin(2\beta_{n1}) + 2\cos(2\beta_{n1}) + 2\beta_{n1}^2 - 2}}$$

for $n=1$, $\beta_{11} = 4.493$, $A_{11} = \frac{6.509}{a^{3/2}}$

for $n=2$, $\beta_{21} = 7.725$, $A_{21} = \frac{11.016}{a^{3/2}}$

$$R_{11}(r) = A_{11}\, j_1\left(\frac{\beta_{11}}{a} r\right) = \frac{6.509}{a^{3/2}}\, j_1\left(\frac{4.493}{a} r\right)$$

$$R_{21}(r) = A_{21}\, j_1\left(\frac{\beta_{21}}{a} r\right) = \frac{11.016}{a^{3/2}}\, j_1\left(\frac{7.725}{a} r\right)$$

$\boxed{l=2}$ $\quad j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x$

$$1 = \int_0^a |A_{n2}|^2 \left| j_2\left(\frac{\beta_{n2}}{a} r\right)\right|^2 r^2\, dr$$

Set $\quad x \equiv \frac{\beta_{n2}}{a} r, \quad r = \frac{a}{\beta_{n2}} x, \quad dr = \frac{a}{\beta_{n2}} dx$

$$1 = \int_0^{\beta_{n2}} |A_{n2}|^2 \cdot \left[\left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x\right]^2 \left(\frac{a}{\beta_{n2}}\right)^3 x^2\, dx$$

$$= |A_{n2}|^2 \left(\frac{a}{\beta_{n2}}\right)^3 \int_0^{\beta_{n2}} \left[\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right]^2 dx$$

$$= |A_{n2}|^2 \cdot \left(\frac{a}{\beta_{n2}}\right)^3 \cdot \frac{\beta_{n2}(12-\beta_{n2}^2)\sin(2\beta_{n2}) + 6(1-\beta_{n2}^2)\cos(2\beta_{n2}) + 2\beta_{n2}^4 - 6\beta_{n2}^2 - 6}{4\beta_{n2}^3}$$

$$A_{n2} = \frac{2\beta_{n2}^3}{a^{3/2}\sqrt{\beta_{n2}(12-\beta_{n2})^2\sin(2\beta_{n2}) + 6(1-\beta_{n2}^2)\cos(2\beta_{n2}) + 2\beta_{n2}^4 - 6\beta_{n2}^2 - 6}}$$

for $n=1$, $\beta_{12} = 5.763$, $A_{12} = 8.542/a^{3/2}$

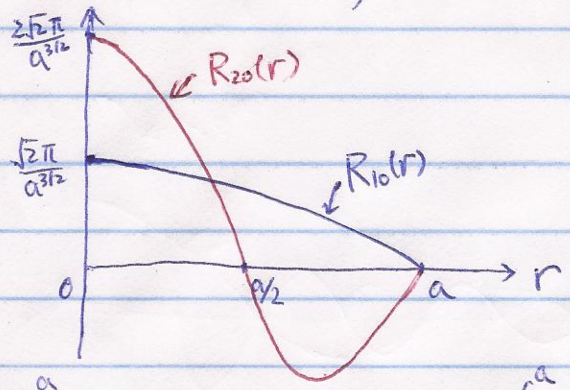
for $n=2$, $\beta_{22} = 9.095$, $A_{22} = 13.102/a^{3/2}$

$$R_{12}(r) = A_{12}\, j_2\left(\frac{\beta_{12}}{a} r\right) = \frac{8.542}{a^{3/2}} j_2\left(\frac{5.763}{a} r\right)$$

$$R_{22}(r) = A_{22}\, j_2\left(\frac{\beta_{22}}{a} r\right) = \frac{13.102}{a^{3/2}} j_2\left(\frac{9.095}{a} r\right)$$

3(b) $R_{10} = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi r}{a}\right)/r \qquad R_{10}(r\to 0) = \sqrt{\frac{2}{a}}\,\frac{\pi}{a} = \frac{\sqrt{2}\pi}{a^{3/2}}$

$R_{20} = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi r}{a}\right)/r \qquad R_{20}(r\to 0) = \sqrt{\frac{2}{a}}\,\frac{2\pi}{a} = \frac{2\sqrt{2}\pi}{a^{3/2}}$



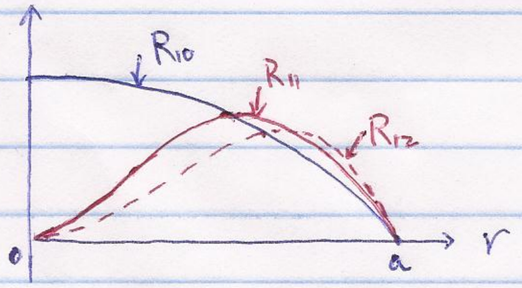
$$\int_0^a R_{10}(r)R_{20}(r)r^2dr = \frac{2}{a}\int_0^a \frac{\sin(\frac{\pi}{a}r)\sin(\frac{2\pi}{a}r)}{r^2}r^2dr$$

$$= \frac{2}{a}\int_0^a \sin\left(\frac{\pi}{a}r\right)\sin\left(\frac{2\pi}{a}r\right)dr = 0$$

Therefore $R_{10}(r)$ and $R_{20}(r)$ are orthogonal to each other.

(c) $R_{10} = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi r}{a}\right)/r$, $R_{11} = \frac{6.509}{a^{3/2}}j_1\left(\frac{4.493}{a}r\right)$.

$R_{12} = \frac{8.542}{a^{3/2}}j_2\left(\frac{5.763}{a}r\right)$



Obviously, none of them are orthogonal to each other because

$\int_0^a \underbrace{R_{10}R_{11}r^2}_{>0}dr > 0 \qquad \int_0^a \underbrace{R_{10}R_{12}r^2}_{>0}dr > 0 \qquad \int_0^a \underbrace{R_{11}R_{12}r^2}_{>0}dr > 0$

The non-orthogonality between R_{10}, R_{11}, R_{12} does not mean that we are using non-orthogonal basis functions because R_{nl} is just the radial part of the whole wavefunction. The whole wave function is

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)\, Y_l^m(\theta,\phi)$$

ψ_{100}, ψ_{11m}, ψ_{12m} are orthogonal to each other because the spherical part of the above three wavefunctions are orthogonal to each other

$$\int_0^{2\pi}\int_0^{\pi} (Y_0^0)^* Y_1^m \sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} (Y_0^0)^* Y_2^m \underbrace{\ }_{\sin\theta} d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} (Y_1^m)^* Y_2^m \underbrace{\ }_{\sin\theta} d\theta\, d\phi = 0$$

[4.33] of Griffiths

$$\int_0^{2\pi}\int_0^{\pi} [Y_l^m(\theta,\phi)]^* [Y_{l'}^{m'}(\theta,\phi)] \sin\theta\, d\theta\, d\phi = \delta_{ll'}\delta_{mm'}$$

4. (a) For small $x \to 0$

$$j_\ell \to \frac{2^\ell \ell!}{(2\ell+1)!} x^\ell \qquad n_\ell \to -\frac{(2\ell)!}{2^\ell \ell!} \frac{1}{x^{\ell+1}}$$

(b) For large $x \to \infty$

$$j_\ell \to \frac{1}{x} \sin\left(x - \ell\frac{\pi}{2}\right) \qquad n_\ell \to -\frac{1}{x} \cos\left(x - \ell\frac{\pi}{2}\right)$$

(c) Spherical Hankel functions

$$h_\ell^{(1)} \equiv j_\ell + i n_\ell \qquad h_\ell^{(2)} \equiv j_\ell - i n_\ell$$

for large $x \to \infty$

$$h_\ell^{(1)} \to \frac{1}{x} \sin\left(x - \ell\frac{\pi}{2}\right) + i\left(-\frac{1}{x}\right) \cos\left(x - \ell\frac{\pi}{2}\right)$$

$$= -i\frac{1}{x}\left[\cos\left(x - \ell\frac{\pi}{2}\right) + i \sin\left(x - \ell\frac{\pi}{2}\right)\right]$$

$$= -i\frac{1}{x} e^{ix} e^{-i\ell\frac{\pi}{2}} = -i\frac{1}{x} e^{ix}\left(e^{-i\frac{\pi}{2}}\right)^\ell = -i\frac{1}{x} e^{ix} (-i)^\ell$$

$$= \frac{1}{x} (-i)^{\ell+1} e^{ix}$$

$$h_\ell^{(2)} \to \frac{1}{x} \sin\left(x - \ell\frac{\pi}{2}\right) + i\frac{1}{x} \cos\left(x - \ell\frac{\pi}{2}\right)$$

$$= i\frac{1}{x}\left[\cos\left(x - \ell\frac{\pi}{2}\right) - i \sin\left(x - \ell\frac{\pi}{2}\right)\right] = i\frac{1}{x} e^{-ix} e^{i\ell\frac{\pi}{2}} = \frac{1}{x} (i)^{\ell+1} e^{-ix}$$

(d) $x \to ix$

$$h_\ell^{(1)} = \frac{1}{ix} (-i)^{\ell+1} e^{-x} \qquad \text{exponentially decreasing as } x \to \infty$$

$$h_\ell^{(2)} = \frac{1}{ix} (i)^{\ell+1} e^{x} \qquad \text{exponentially increasing as } x \to \infty$$

5. The energy level of a hydrogen atom is given in Griffiths [4.70]

(a) $$E_n = -\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}, \quad n = 1, 2, 3, \cdots$$

where m_e is the mass of the electron. To be more precise, m_e is actually the reduced mass of the electron with respect to the proton (nucleus), which has been assumed to be motionless. The relation between the reduced mass, μ, and the actual mass, m_e, is $$\frac{1}{\mu} = \frac{1}{m_e} + \frac{1}{M},$$

where M is the mass of the nucleus. For hydrogen, the nucleus contains just a proton and $M_H \approx 1836.15 m_e$. For deuterium, the nucleus contains a proton and a neutron, $M_D \approx 1836.15 m_e + 1838.68 m_e = 3674.83 m_e$

For hydrogen, $$\mu_H = \frac{1}{\frac{1}{m_e} + \frac{1}{M_H}} = \frac{1}{1 + \frac{1}{1836.15}} m_e = 0.99946\, m_e$$

For deuterium, $$\mu_D = \frac{1}{\frac{1}{m_e} + \frac{1}{M_D}} = \frac{1}{1 + \frac{1}{3674.83}} m_e = 0.99973\, m_e$$

The difference between the ground state energies of the hydrogen and of the deuterium is thus

$$E^D - E^H = -\left[\frac{\mu_D}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right] + \left[\frac{\mu_H}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]$$

$$= (-0.99973 + 0.99946)\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]$$

$$= -0.00027 \times 13.6\,\text{eV}$$

$$= -0.00367\,\text{eV} = -3.67\,\text{meV}$$

The ground state of deuterium is 3.67 meV below that of hydrogen.

(b) For a positronium, $M = m_e$ (the mass of e^+)

$$\mu = \frac{1}{\frac{1}{m_e} + \frac{1}{M}} = \frac{1}{\frac{1}{m_e} + \frac{1}{m_e}} = \frac{1}{2} m_e$$

The ground state energy is

$$E^{\text{positronium}} = -\left[\frac{\mu}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} \quad \text{with } n=1 \text{ for ground state}$$

$$= -\frac{1}{2}\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right] = -\frac{13.6\,\text{eV}}{2} = -6.8\,\text{eV}$$

(c) For a muonic hydrogen $M = 206 m_e$

$$\mu = \frac{1}{\frac{1}{m_e} + \frac{1}{M}} = \frac{1}{1+\frac{1}{206}} m_e = \frac{206}{207} m_e = 0.9952\, m_e$$

the ground state energy is $E_1 = -0.9952 \times 13.6\,\text{eV} = -13.53\,\text{eV}$

the 1st excited state energy is $E_2 = -0.9952 \times \frac{13.6\,\text{eV}}{4} = -3.38\,\text{eV}$

The transition energy between $n=1$ and $n=2$ is

$$\Delta E = E_2 - E_1 = 10.15\,\text{eV} = 122.15\,\text{nm in wavelength}$$

(d) From Griffiths [4.52] to [4.70], changing the nuclear charge from 1 to Z basically only changes [4.55], where ρ_0 is now

$$\rho_0 = \frac{me^2 Z}{2\pi\epsilon_0\hbar^2 k} \Rightarrow k = \frac{me^2 Z}{2\pi\epsilon_0\hbar^2\rho_0}$$

this will affect [4.69], which is now

$$E = -\frac{\hbar^2 k^2}{2m} = -\frac{me^4 Z^2}{8\pi^2\epsilon_0^2\hbar^2\rho_0^2} = -\left[\frac{m}{2\hbar^2}\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}$$

So the energy levels will scale as Z^2

(e) For an exciton with $M_{hole} = 0.5 m_e$ and $M_{electron} = 0.5 m_e$,

the reduced mass

$$\mu = \frac{1}{\frac{1}{M_{hole}} + \frac{1}{M_{electron}}} = \frac{1}{\frac{1}{0.5} + \frac{1}{0.5}} m_e = \frac{1}{4} m_e$$

The Coulomb potential is now

$$V(r) = -\frac{e^2}{4\pi\epsilon}\frac{1}{r} \quad \text{with } \epsilon = 16\epsilon_0$$

So the Coulomb potential is 16 times weaker than in vacuum.

The ground state energy of this exciton is now

$$E_1^{exciton} = -\left[\frac{\mu}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon}\right)^2\right] = -\frac{1}{4}\cdot\frac{1}{16^2}\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]$$

$$= -\frac{1}{1024}\cdot 13.6\,\text{eV} = -0.01328\,\text{eV} = -13.28\,\text{meV}$$

6. (a) $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$

where $V(x,y,z) = \begin{cases} 0 & \text{if } 0\le x\le a \text{ and } 0\le y\le a \text{ and } 0\le z\le a \\ \infty & \text{otherwise} \end{cases}$.

(b) We know for a 1D infinite square well, the energy levels are

$$E_n = \frac{\pi^2\hbar^2}{2ma^2}n^2, \quad n = 1, 2, \cdots$$

This applies for each dimension of the 3D box (infinite square well). The total energy is determind by the n numbers along each dimension, i.e., n_x, n_y, n_z.

(b)&(c)
$$E_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2+n_y^2+n_z^2), \quad n_x, n_y, n_z = 1,2,3,\cdots$$

The lowest 5 eigenenergies are

1. $E_{111} = \frac{\pi^2\hbar^2}{2ma^2}\cdot 3$ — no degeneracy
2. $E_{211} = E_{121} = E_{112} = \frac{\pi^2\hbar^2}{2ma^2}\cdot 6$ — 3-fold degeneracy
3. $E_{221} = E_{122} = E_{212} = \frac{\pi^2\hbar^2}{2ma^2}\cdot 9$ — 3-fold degeneracy
4. $E_{311} = E_{131} = E_{113} = \frac{\pi^2\hbar^2}{2ma^2}\cdot 11$ — 3-fold degeneracy
5. $E_{222} = \frac{\pi^2\hbar^2}{2ma^2}\cdot 12$ — no degeneracy

~~(d)~~

7. (a) The system is in the state with $L_z=1$, or the $\begin{pmatrix}1\\0\\0\end{pmatrix}$ state. If L_x is measured, the possible outcomes are the eigenvalues of L_x, which can be obtained as

$$\begin{vmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{vmatrix} = -\lambda^3+\lambda = -\lambda(\lambda+1)(\lambda-1)=0 \Rightarrow \lambda=1,0,-1$$

For $\lambda=1$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix} \Rightarrow \begin{pmatrix}\frac{1}{\sqrt{2}}b\\ \frac{1}{\sqrt{2}}(a+c)\\ \frac{1}{\sqrt{2}}b\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$\Rightarrow \begin{cases}\frac{1}{\sqrt{2}}b=a\\ \frac{1}{\sqrt{2}}(a+c)=b\\ \frac{1}{\sqrt{2}}b=c\end{cases}$$

together with $a^2+b^2+c^2=1$ we get $a=c=\frac{1}{2}$, $b=\frac{1}{\sqrt{2}}$

For $\lambda=0$

$$\begin{pmatrix}\frac{1}{\sqrt{2}}b\\ \frac{1}{\sqrt{2}}(a+c)\\ \frac{1}{\sqrt{2}}b\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{cases}b=0\\a+c=0\\a^2+b^2+c^2=1\end{cases} \Rightarrow \begin{cases}a=\frac{1}{\sqrt{2}}\\b=0\\c=-\frac{1}{\sqrt{2}}\end{cases}$$

For $\lambda=-1$

$$\begin{pmatrix}\frac{1}{\sqrt{2}}b\\ \frac{1}{\sqrt{2}}(a+c)\\ \frac{1}{\sqrt{2}}b\end{pmatrix}=\begin{pmatrix}-a\\-b\\-c\end{pmatrix} \Rightarrow \begin{cases}a=c\\b=-\sqrt{2}a\\a^2+b^2+c^2=1\end{cases} \Rightarrow \begin{cases}a=\frac{1}{2}\\b=-\frac{1}{\sqrt{2}}\\c=\frac{1}{2}\end{cases}$$

The initial state of the system, $\begin{pmatrix}1\\0\\0\end{pmatrix}$, can be written as a linear combination of the eigenstates of L_x:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = C_1\begin{pmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix} + C_2\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix} + C_3\begin{pmatrix}\frac{1}{2}\\-\frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix}$$

where $C_1 = \left(\frac{1}{2}\ \frac{1}{\sqrt{2}}\ \frac{1}{2}\right)\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{2}$

$$c_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$c_3 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}$$

So possible outcomes and the corresponding probabilities when L_x is measured are
1 : probability $\frac{1}{4}$
0 : probability $\frac{1}{2}$
-1 : probability $\frac{1}{4}$

The expectation value is $\frac{1}{4}\cdot 1 + \frac{1}{2}\cdot 0 + \frac{1}{4}(-1) = 0$

(b) i) If L_z^2 is measured and the result is $+1$, then the state after the measurement should be an eigenstate of L_z^2 with eigenvalue $+1$.

$$L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow L_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues of L_z^2 can be obtained as

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = -\lambda(1-\lambda)(1-\lambda) = 0 \quad \Rightarrow \lambda = 0, \underbrace{+1}_{\text{two-fold degenerate}}$$

For $\lambda = +1$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow b = 0$$

As long as $b=0$, any state $\begin{pmatrix} a \\ 0 \\ c \end{pmatrix}$ is an eigenstate of L_z^2 with eigenvalue $+1$.

Because there are two eigenstates with eigenvalue $+1$, we need to find the other eigenstate. In this situation, the second eigenstate is usually assumed to be orthogonal to the first one, which is $\begin{pmatrix} a \\ 0 \\ c \end{pmatrix}$ here.

Therefore the 2nd eigenstate can be chosen as $\begin{pmatrix} c \\ 0 \\ -a \end{pmatrix}$.

And also we assume that $a^2+c^2=1$, i.e., the two eigenstates are normalized.

The probability to be in either the $\begin{pmatrix} a \\ 0 \\ c \end{pmatrix}$ state or the $\begin{pmatrix} c \\ 0 \\ -a \end{pmatrix}$ state is

$$P = \left| (a\ 0\ c) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2 + \left| (c\ 0\ -a) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2$$

$$= \left| \tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}c \right|^2 + \left| \tfrac{1}{2}c - \tfrac{1}{\sqrt{2}}a \right|^2$$

$$= \tfrac{1}{4}a^2 + \tfrac{1}{2}c^2 + \tfrac{1}{\sqrt{2}}ac + \tfrac{1}{4}c^2 + \tfrac{1}{2}a^2 - \tfrac{1}{\sqrt{2}}ac$$

$$= \tfrac{1}{4}(a^2+c^2) + \tfrac{1}{2}(a^2+c^2) = \tfrac{3}{4}$$

ii) $$\Psi_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

If L_z is measured, possible outcomes are

1 : prob $\frac{1}{4}$ $\quad$ 0 : prob $\frac{1}{4}$ $\quad$ -1 : prob $\frac{1}{2}$