HW 3

1. Griffiths 4.27 page 177

(a) $1 = |\chi|^2 = \chi^\dagger \chi = A^*(-3i \quad 4) A\begin{pmatrix} 3i \\ 4 \end{pmatrix} = |A|^2 \cdot 25 \Rightarrow A = \frac{1}{5}$

(b) $\langle S_x \rangle = \chi^\dagger S_x \chi = \frac{1}{5}(-3i \quad 4) \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{5}\begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{1}{25} \cdot \frac{\hbar}{2} \cdot (-3i \quad 4)\begin{pmatrix} 4 \\ 3i \end{pmatrix}$

$= 0$

$\langle S_y \rangle = \chi^\dagger S_y \chi = \frac{1}{5}(-3i \quad 4) \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{5}\begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar}{50}(-3i \quad 4)\begin{pmatrix} -4i \\ -3 \end{pmatrix}$

$= \frac{\hbar}{50}(-12-12) = -\frac{12}{25}\hbar$

$\langle S_z \rangle = \chi^\dagger S_z \chi = \frac{1}{5}(-3i \quad 4) \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{5}\begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar}{50}(-3i \quad 4)\begin{pmatrix} 3i \\ -4 \end{pmatrix}$

$= \frac{\hbar}{50}(9-16) = -\frac{7}{50}\hbar$

(c) $S_x^2 = S_y^2 = S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \chi^\dagger \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \chi$

$= \frac{1}{5}(-3i \quad 4) \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{5}\begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar^2}{100}(-3i \quad 4)\begin{pmatrix} 3i \\ 4 \end{pmatrix}$

$= \frac{\hbar^2}{100} \cdot 25 = \frac{\hbar^2}{4}$

$\sigma_{S_x} = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2}$

$\sigma_{S_y} = \sqrt{\langle S_y^2 \rangle - \langle S_y \rangle^2} = \sqrt{\frac{\hbar^2}{4} - \frac{144}{625}\hbar^2} = \frac{7}{50}\hbar$

$\sigma_{S_z} = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2} = \sqrt{\frac{\hbar^2}{4} - \frac{49}{2500}\hbar^2} = \frac{12}{25}\hbar$

(d) $\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}\frac{7}{50}\hbar = \frac{7}{100}\hbar^2 = \frac{\hbar}{2}|\langle S_z \rangle|$ minimum uncertai

$\sigma_{S_y}\sigma_{S_z} = \frac{7}{50}\hbar \cdot \frac{12}{25}\hbar = \frac{42}{625}\hbar^2 > \frac{\hbar}{2}|\langle S_x \rangle| = 0$

$\sigma_{S_z}\sigma_{S_x} = \frac{12}{25}\hbar \cdot \frac{\hbar}{2} = \frac{6}{25}\hbar^2 = \frac{\hbar}{2}|\langle S_y \rangle|$ minimum uncertaint

2. Griffiths 4.29 page 177

(a). $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Suppose eigenstate (eigenspinor) $\begin{pmatrix} a \\ b \end{pmatrix}$ and eigenvalue λ

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$$, a, b are complex in general

$$\frac{\hbar}{2}\begin{pmatrix} -ib \\ ia \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{cases} \lambda a = -\frac{i\hbar}{2} b & (1) \\ \lambda b = \frac{i\hbar}{2} a & (2) \end{cases}$$

suppose the eigenstate $\begin{pmatrix} a \\ b \end{pmatrix}$ is normalized : $|a|^2 + |b|^2 = 1$ (3)

From (1), (2), & (3) one gets

$\lambda = \pm \frac{\hbar}{2}$ (as expected)

$a = \frac{1}{\sqrt{2}}$, $\boxed{b = ia = \frac{i}{\sqrt{2}} \ (\text{if } \lambda = \frac{\hbar}{2})}$ or $\boxed{b = -ia = -\frac{i}{\sqrt{2}} \ (\text{if } \lambda = -\frac{\hbar}{2})}$

So the eigenstates are

$\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ eigenvalue $\frac{\hbar}{2}$

$\chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ eigenvalue $-\frac{\hbar}{2}$

(b) If we measure S_y on a particle in the general state $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$, we will get one of the two eigenvalues of S_y, either $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$.

The probability of getting the value $\frac{\hbar}{2}$ is

$$\left|(\chi_+^{(y)})^+ \chi\right|^2 = \left|\frac{1}{\sqrt{2}}(1 \ -i)\begin{pmatrix} a \\ b \end{pmatrix}\right|^2 = \frac{1}{2}|a - ib|^2$$

the probability of getting the value $-\frac{\hbar}{2}$ is

$$\left|\left(\chi_-^{(y)}\right)^\dagger \chi\right|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\right|^2 = \frac{1}{2}|a+ib|^2$$

the sum of the two probabilities is

$$\frac{1}{2}|a-ib|^2 + \frac{1}{2}|a+ib|^2$$

$$= \frac{1}{2}(a^* + ib^*)(a-ib) + \frac{1}{2}(a^* - ib^*)(a+ib)$$

$$= \frac{1}{2}\left[\,|a|^2 + ib^*a - ia^*b + |b|^2 + |a|^2 - ib^*a + ia^*b + |b|^2\right]$$

$$= |a|^2 + |b|^2 = 1$$

(c) No matter which eigenstate of S_y you get, the eigenvalue of S_y^2 is $\frac{\hbar^2}{4}$. So you will get this value with probability 1.

3. 4.30 Griffiths page 178

$$S_r = S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z \cos\theta$$

$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\phi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta\right]$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

Suppose normalized eigenstate $\begin{pmatrix} a \\ b \end{pmatrix}$ and eigenvalue $\left(\lambda\frac{\hbar}{2}\right)$

then $\frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\lambda\begin{pmatrix} a \\ b \end{pmatrix}$

$$\Rightarrow \quad \cancel{\frac{\hbar}{2}}\left(a\cos\theta + b e^{-i\phi}\sin\theta\right) = \cancel{\frac{\hbar}{2}}\lambda a \qquad (1)$$

$$\cancel{\frac{\hbar}{2}}\left(a e^{i\phi}\sin\theta - b\cos\theta\right) = \cancel{\frac{\hbar}{2}}\lambda b \qquad (2)$$

normalization condition $\quad |a|^2 + |b|^2 = 1 \qquad (3)$

From (1) $\quad b = (\lambda - \cos\theta)\, a \cdot \frac{e^{i\phi}}{\sin\theta} \qquad (4)$

From (2) $\quad b = \frac{a e^{i\phi}\sin\theta}{\lambda + \cos\theta} \qquad (5)$

From (4) & (5) $\quad (\lambda - \cos\theta)\cdot a \cdot \frac{e^{i\phi}}{\sin\theta} = \frac{a e^{i\phi}\sin\theta}{\lambda + \cos\theta}$

$$a \cdot e^{i\phi} \cdot \left(\lambda^2 - \cos^2\theta - \sin^2\theta\right) = 0$$

$$a e^{i\phi}(\lambda^2 - 1) = 0$$

$$\Rightarrow \quad \lambda = \pm 1$$

Therefore the eigenvalues of S_r are $\left(\lambda\frac{\hbar}{2}\right) = \pm\frac{\hbar}{2}$

(as expected)

If $\lambda = 1$, from (5) $b = \dfrac{a e^{i\phi} \sin\theta}{1+\cos\theta}$

use (3) $|a|^2 + |b|^2 = |a|^2 + |a|^2 \left(\dfrac{\sin\theta}{1+\cos\theta}\right)^2 = 1$

$$1 = |a|^2 \cdot \left(1 + \frac{\sin^2\theta}{(1+\cos\theta)^2}\right) = |a|^2 \cdot \frac{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}{(1+\cos\theta)^2}$$

$$= |a|^2 \cdot \frac{2(1+\cos\theta)}{(1+\cos\theta)^2} = |a|^2 \cdot \frac{2}{1+\cos\theta}$$

$$\Rightarrow |a|^2 = \frac{1+\cos\theta}{2} = \cos^2\left(\frac{\theta}{2}\right)$$

We can take $a = \cos(\theta/2)$ then

$$b = \frac{\cos(\theta/2)\, e^{i\phi} \sin\theta}{1+\cos\theta} = \frac{\cos(\theta/2)\, e^{i\phi} \cdot 2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = e^{i\phi} \sin(\theta/2)$$

then the eigenstate is $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}$

If $\lambda = -1$, from (5) $b = \dfrac{a e^{i\phi} \sin\theta}{-1+\cos\theta}$

use (3) $|a|^2 + |b|^2 = |a|^2 + |a|^2 \cdot \dfrac{\sin^2\theta}{(1-\cos\theta)^2} = 1$

$$1 = |a|^2 \left(1 + \frac{\sin^2\theta}{(1-\cos\theta)^2}\right) = |a|^2 \cdot \frac{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}{(1-\cos\theta)^2}$$

$$= |a|^2 \cdot \frac{2(1-\cos\theta)}{(1-\cos\theta)^2} = |a|^2 \cdot \frac{2}{1-\cos\theta}$$

$$\Rightarrow |a|^2 = \frac{1-\cos\theta}{2} = \sin^2\left(\frac{\theta}{2}\right)$$

We can take $a = \sin(\theta/2)$ then

$$b = \frac{\sin(\theta/2)\, e^{i\phi} \sin\theta}{-1+\cos\theta} = \frac{\sin(\theta/2)\, e^{i\phi}\, 2\sin(\theta/2)\cos(\theta/2)}{-2\sin^2(\theta/2)} = -e^{i\phi}\cos(\theta/2)$$

then the eigenstate is $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix}$

4. Griffiths 4.31 page 178

For a spin-1 particle, $s=1$, $m=-1,0,1$

S_z has three eigenstates $\chi_+ = |1,1\rangle$, $\chi_0 = |1,0\rangle$, $\chi_- = |1,-1\rangle$

In vector form

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From [4.136] $S_+\chi_+ = 0 \qquad S_+\chi_0 = \sqrt{2}\hbar\chi_+ \qquad S_+\chi_- = \sqrt{2}\hbar\chi_0$

$S_-\chi_+ = \sqrt{2}\hbar\chi_0 \qquad S_-\chi_0 = \sqrt{2}\hbar\chi_- \qquad S_-\chi_- = 0$

Suppose $S_+ = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$

$S_+\chi_+ = 0 \Rightarrow C_{11} = C_{21} = C_{31} = 0$

$S_+\chi_0 = \sqrt{2}\hbar\chi_+ \Rightarrow C_{12} = \sqrt{2}\hbar, \ C_{22} = C_{32} = 0$

$S_+\chi_- = \sqrt{2}\hbar\chi_0 \Rightarrow C_{13} = 0, \ C_{23} = \sqrt{2}\hbar, \ C_{33} = 0$

Therefore $S_+ = \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, Similarly $S_- = \sqrt{2}\hbar\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{2i}(S_+ - S_-) = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Suppose $S_z = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$

$$\left.\begin{array}{l} S_z\chi_+ = \hbar\chi_+ \Rightarrow d_{11} = \hbar, \ d_{21} = 0, \ d_{31} = 0 \\ S_z\chi_0 = 0 \Rightarrow d_{12} = d_{22} = d_{32} = 0 \\ S_z\chi_- = -\hbar\chi_- \Rightarrow d_{13} = d_{23} = 0, \ d_{33} = -\hbar \end{array}\right\} \Rightarrow S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

5. In the $|+\rangle$ and $|-\rangle$ basis states

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

the eigenvalue equation of S_x is.

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm\frac{\hbar}{2} \text{ as expected}$$

Suppose the eigenstate of S_x is $\begin{pmatrix} a \\ b \end{pmatrix}$

For $\lambda = \frac{\hbar}{2}$, $\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow b = a$

normalization condition $|a|^2 + |b|^2 = 1$, $|a|^2 = \frac{1}{2}$

one can pick $a = \frac{1}{\sqrt{2}}$, then $b = \frac{1}{\sqrt{2}}$

So $\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda = -\frac{\hbar}{2}$, $\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow b = -a$

normalization condition $|a|^2 + |b|^2 = 1$, $|a|^2 = \frac{1}{2}$

one can pick $a = \frac{1}{\sqrt{2}}$, then $b = -\frac{1}{\sqrt{2}}$

So $\chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Therefore in the $|+\rangle$ and $|-\rangle$ basis states the eigenstates of S_x can be expressed as

$$|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

$$|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

(a) In the basis set of $|+\rangle_x$ and $|-\rangle_x$

let $P_{+x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$|+\rangle_x$ in the basis set of $|+\rangle_x$ and $|-\rangle_x$

By definition $P_{+x}|+\rangle_x = 1\cdot|+\rangle_x \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow a=1,\ c=0$

and $P_{+x}|-\rangle_x = 0\,|-\rangle_x \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow b=0,\ d=0$

$|-\rangle_x$ in the basis set of $|+\rangle_x$ and $|-\rangle_x$

therefore $P_{+x} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in the $|+\rangle_x$ and $|-\rangle_x$ basis

(b) Suppose $P_{+x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the basis set of $|+\rangle$ and $|-\rangle$

$|+\rangle_x$ in the basis set of $|+\rangle$ and $|-\rangle$

$P_{+x}|+\rangle_x = 1\cdot|+\rangle_x \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$$\Rightarrow \begin{cases} \frac{1}{\sqrt{2}}(a+b) = \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}(c+d) = \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \begin{cases} a+b=1 \\ c+d=1 \end{cases} \quad (1)$$

$|-\rangle_x$ in the basis set of $|+\rangle$ and $|-\rangle$

$P_{+x}|-\rangle_x = 0\cdot|-\rangle_x \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{cases} \frac{1}{\sqrt{2}}(a-b) = 0 \\ \frac{1}{\sqrt{2}}(c-d) = 0 \end{cases} \Rightarrow \begin{cases} a=b \\ c=d \end{cases} \quad (2)$$

From (1) & (2) we get $a=b=c=d=\frac{1}{2}$

$P_{+x} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ in the $|+\rangle$ and $|-\rangle$ basis

(c) In the $|+\rangle_x$ and $|-\rangle_x$ basis, $|+\rangle_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$|+\rangle_x \langle +|_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_{+x}$$ as shown in (a)

In the $|+\rangle$ and $|-\rangle$ basis, $|+\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$$|+\rangle_x \langle +|_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = P_{+x}$$ as shown in (b)