# HW #2

1. If $\phi(x)$ is a solution of the Schrödiger Eq.

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\phi(x) = E\phi(x)$$

Change $x \longrightarrow -x$

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(-x)\right]\phi(-x) = E\phi(-x)$$

b/c $V(-x) = V(x)$

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\phi(-x) = E\phi(-x)$$

Therefore $\phi(-x)$ is also a solution of the Schrödiger Eq.
A linear combination of $\phi(x)$ and $\phi(-x)$

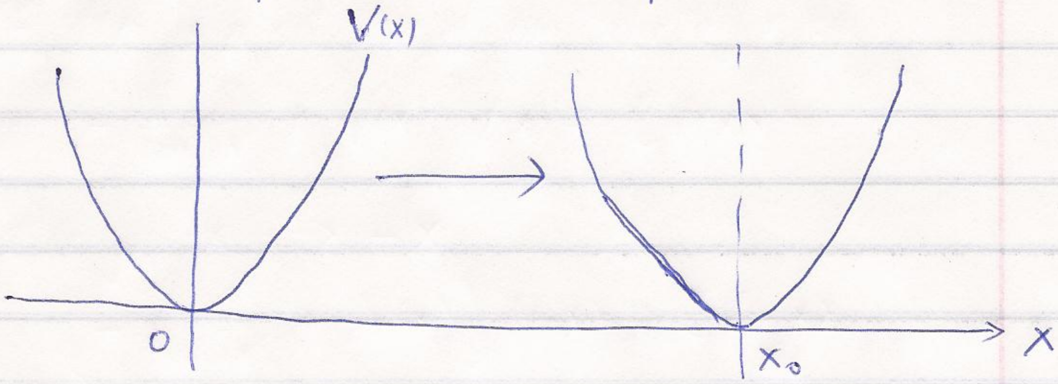
$\phi(x) + \phi(-x)$ (Even)

or $\phi(x) - \phi(-x)$ (Odd)

is also a solution of the Schrödinger Eq.

Note that $\phi(x)$, $\phi(-x)$, $\phi(x)+\phi(-x)$, $\phi(x)-\phi(-x)$ have the same energy $E$.

2. $V(x)$ is symmetric with respect to the $x = x_0$ line.




The energy of the particle does not depend on the location of the potential. So the ground state energy of the new potential is still $\hbar\omega/2$.

The wavefunction will be moved with the potential

So $x \longrightarrow x - x_0$

$$\psi_0'(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}(x-x_0)^2}$$

3. $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$ $\qquad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$

$H = T + V$

$T = \frac{p^2}{2m} = \frac{1}{2m}(-1)\frac{\hbar m\omega}{2}(a_+ - a_-)^2$

$= -\frac{\hbar\omega}{4}(a_+a_+ - a_-a_+ - a_+a_- + a_-a_-)$

$\langle n|T|n\rangle = -\frac{\hbar\omega}{4}\langle n|a_+a_+ - a_-a_+ - a_+a_- + a_-a_-|n\rangle$

b/c $\langle n|a_+a_+|n\rangle = \langle n|a_-a_-|n\rangle = 0$

$\langle n|a_-a_+|n\rangle = n+1$

$\langle n|a_+a_-|n\rangle = n$

$\langle n|T|n\rangle = -\frac{\hbar\omega}{4}(-2n-1) = \frac{\hbar\omega}{4}(2n+1)$

$V = \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}(a_+ + a_-)^2$

$= \frac{1}{4}\hbar\omega(a_+a_+ + a_+a_- + a_-a_+ + a_-a_-)$

$\langle n|V|n\rangle = \frac{1}{4}\hbar\omega(2n+1)$

So $\langle T\rangle = \langle V\rangle$ for any $|n\rangle$

Eigenstates of $H$ | Eigenstates of $H'$

4.

$\frac{9}{2}\hbar\omega$ —— $\phi_4$

$\frac{7}{2}\hbar\omega$ —— $\phi_3$

$\frac{5}{2}\hbar\omega$ —— $\phi_2$

$\frac{3}{2}\hbar\omega$ —— $\phi_1$

$\frac{1}{2}\hbar\omega$ —— $\phi_0$

$5\hbar\omega$ —— $\phi_2'$

$3\hbar\omega$ —— $\phi_1'$

$\hbar\omega$ —— $\phi_0'$

The Hamiltonian of the particle changes into a new one ($H'$) when the spring constant changes. So any measurement on energy made after the spring constant changes can only give a value of one of the eigenvalues of the new Hamiltonian $H'$, i.e., $\hbar\omega$, $3\hbar\omega$, $5\hbar\omega$, ...
The probability to give $\frac{1}{2}\hbar\omega$ is zero.

The wave function of the particle is still $\phi_0(x)$, which can be expressed as a linear combination of the new eigenstates

$$\phi_0(x) = \sum_{n=0}^{\infty} C_n \phi_n'(x)$$

The probability of finding the particle in the new ground state with energy $\hbar\omega$ is $|C_0|^2$

$$C_0 = \langle \phi_0'(x) | \phi_0(x) \rangle = \int_{-\infty}^{\infty} \left(\frac{m\omega'}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega'}{2\hbar}x^2} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} dx$$

$$= \left(\frac{m\omega'}{\pi\hbar}\right)^{\frac{1}{4}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-\left(\frac{m\omega'}{2\hbar} + \frac{m\omega}{2\hbar}\right)x^2} dx$$

$$= \left(\frac{\sqrt{2}\, m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{3m\omega}{2\hbar}x^2} dx = \left(\frac{\sqrt{2}\, m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{2\pi\hbar}{3m\omega}\right)^{\frac{1}{2}} = \sqrt{\frac{2\sqrt{2}}{3}}$$

$|C_0|^2 = 2\sqrt{2}/3 = 0.943$

5. The harmonic oscillator is in a superposition of the ground state and the 1st excited state

$$\psi(x,t) = c_0 \psi_0(x) e^{-iE_0 t/\hbar} + c_1 \psi_1(x) e^{-iE_1 t/\hbar}$$

$$|c_0|^2 = |c_1|^2 = \frac{1}{2} \Rightarrow c_0 = \frac{e^{i\theta_0}}{\sqrt{2}}, \quad c_1 = \frac{e^{i\theta_1}}{\sqrt{2}}, \quad E_0 = \frac{1}{2}\hbar\omega, E_1 = \frac{3}{2}\hbar\omega$$

$$\langle p \rangle = \langle \psi(x,t) | p | \psi(x,t) \rangle$$

$$= |c_0|^2 \langle \psi_0 | p | \psi_0 \rangle + |c_1|^2 \langle \psi_1 | p | \psi_1 \rangle$$
$$+ c_0^* c_1 e^{i(E_0 - E_1)t/\hbar} \langle \psi_0 | p | \psi_1 \rangle + c_0 c_1^* e^{i(E_1 - E_0)t/\hbar} \langle \psi_1 | p | \psi_0 \rangle$$

$$E_1 - E_0 = \hbar\omega, \quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

$$\langle \psi_0 | p | \psi_0 \rangle = \langle \psi_1 | p | \psi_1 \rangle = 0$$

$$\langle \psi_0 | p | \psi_1 \rangle = i\sqrt{\frac{\hbar m\omega}{2}} \langle \psi_0 | a_+ - a_- | \psi_1 \rangle = -i\sqrt{\frac{\hbar m\omega}{2}}$$

$$\langle \psi_1 | p | \psi_0 \rangle = \langle \psi_0 | p | \psi_1 \rangle^* = i\sqrt{\frac{\hbar m\omega}{2}}$$

So, $\langle p \rangle = c_0^* c_1 e^{-i\omega t}\left(-i\sqrt{\frac{\hbar m\omega}{2}}\right) + c_0 c_1^* e^{i\omega t} i\sqrt{\frac{\hbar m\omega}{2}}$

$$= \frac{e^{-i\theta_0}}{\sqrt{2}} \frac{e^{i\theta_1}}{\sqrt{2}} e^{-i\omega t}\left(-i\sqrt{\frac{\hbar m\omega}{2}}\right) + \frac{e^{i\theta_0}}{\sqrt{2}} \frac{e^{-i\theta_1}}{\sqrt{2}} e^{i\omega t} i\sqrt{\frac{\hbar m\omega}{2}}$$

$$= \frac{i}{2}\sqrt{\frac{\hbar m\omega}{2}} \underbrace{\left(e^{i(\theta_0 - \theta_1 + \omega t)} - e^{-i(\theta_0 - \theta_1 + \omega t)}\right)}_{= 2i\sin(\theta_0 - \theta_1 + \omega t)}$$

$$= -\sqrt{\frac{\hbar m\omega}{2}} \sin(\theta_0 - \theta_1 + \omega t)$$

$$= \sqrt{\frac{\hbar m\omega}{2}} \sin(\theta_1 - \theta_0 - \omega t)$$

So the maximum value of $\langle p \rangle$ is $\sqrt{\frac{\hbar m \omega}{2}}$

If we assume this maximum value to occur at $t=0$

then $\sin(\theta_1 - \theta_0) = 1$, $\theta_1 - \theta_0 = \frac{\pi}{2}$, $\theta_1 = \theta_0 + \frac{\pi}{2}$

$$\psi(x,t) = \frac{e^{i\theta_0}}{\sqrt{2}} \psi_0(x) e^{-iE_0 t/\hbar} + \frac{e^{i\theta_1}}{\sqrt{2}} \psi_1(x) e^{-iE_1 t/\hbar}$$

$$= \frac{e^{i\theta_0}}{\sqrt{2}} \left[ \psi_0(x) e^{-iE_0 t/\hbar} + e^{i\frac{\pi}{2}} \psi_1(x) e^{-iE_1 t/\hbar} \right]$$

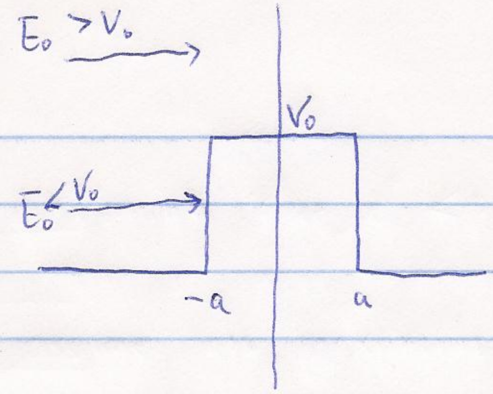
$$= \frac{e^{i\theta_0}}{\sqrt{2}} \left[ \psi_0(x) e^{-iE_0 t/\hbar} + i \psi_1(x) e^{-iE_1 t/\hbar} \right]$$

$\theta_0$ can be any value, so let $\theta_0 = 0$ for simplicity

$$\psi(x,t) = \frac{1}{\sqrt{2}} \left[ \psi_0(x) e^{-iE_0 t/\hbar} + i \psi_1(x) e^{-iE_1 t/\hbar} \right]$$

where $E_0 = \frac{\hbar\omega}{2}$, $E_1 = \frac{3\hbar\omega}{2}$

6.




[2.169] Griffiths

$$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)$$

but this is for a square well with potential $-V_0$ $(-a \le x \le a)$

For a barrier with potential $+V_0$, we should switch the sign of $V_0$

$$T^{-1} = 1 + \frac{V_0^2}{4E(E-V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E-V_0)}\right)$$

This trasmission coefficient formula is good for $\underline{E > V_0}$

For $\underline{E < V_0}$, $E - V_0 < 0$, $\sin\left(\frac{2a}{\hbar}\sqrt{2m(E-V_0)}\right) = \sin\left(i\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right)$

use the relation $\sin(i\theta) = i\sinh(\theta)$

$$\sin\left(i\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right) = i\sinh\left(\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right)$$

$$T^{-1} = 1 + \frac{V_0^2}{4E(V_0-E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right)$$

For $\underline{E = V_0}$

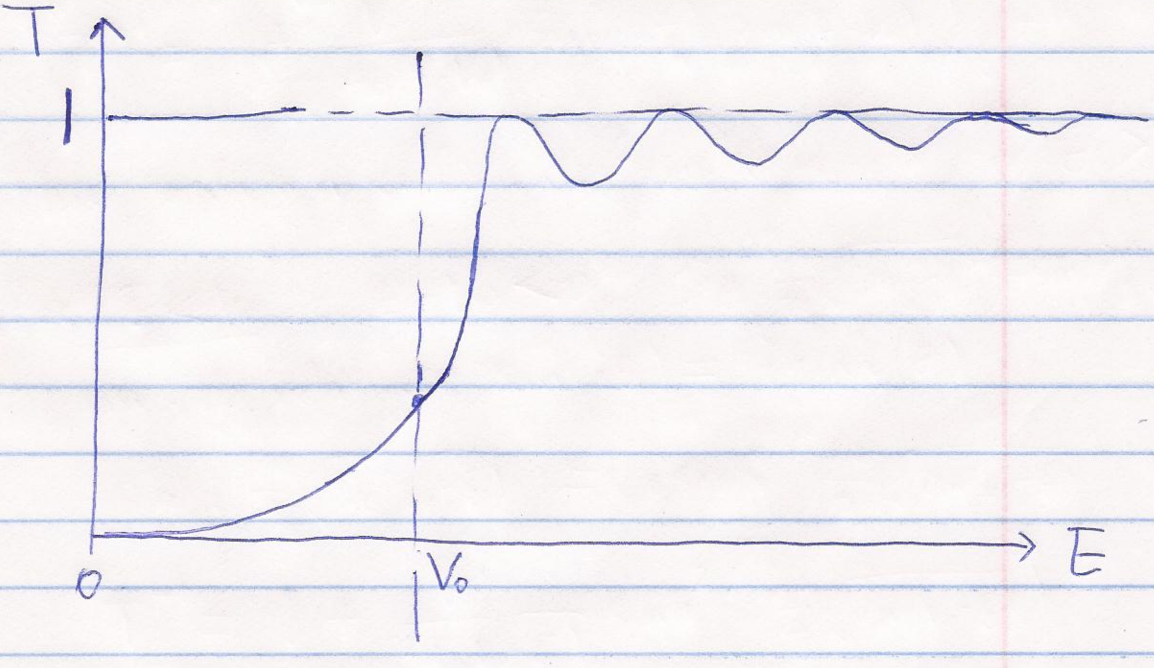
recall $\lim_{X \to 0} \sin X = X$

$$\lim_{(E-V_0)\to 0} \sin\left(\frac{2a}{\hbar}\sqrt{2m(E-V_0)}\right) = \frac{2a}{\hbar}\sqrt{2m(E-V_0)}$$

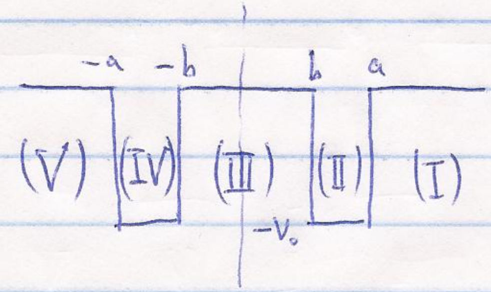
$$\lim_{(E-V_0)\to 0} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E-V_0)}\right) = \frac{4a^2}{\hbar^2}\cdot 2m\cdot(E-V_0)$$

$$\lim_{(E-V_0)\to 0} T^{-1} = 1 + \frac{V_0^2}{4E(E-V_0)}\cdot\frac{4a^2}{\hbar^2}2m(E-V_0)$$

$$= 1 + \frac{2mV_0}{\hbar^2}a^2$$



7.




<u>Even solution</u>

$$\psi(x) = \begin{cases} A e^{-\alpha x} & x \in (\mathrm{I}) \\ B\sin(qx) + C\cos(qx) & x \in (\mathrm{II}) \\ D e^{-\alpha x} + F e^{+\alpha x} & x \in (\mathrm{III}) \\ \quad F = D \text{ even} & \\ \psi(-x) \quad F = -D \text{ odd} & x < 0 \end{cases}$$

In region (III), $\psi(x) = \psi(-x)$ (Even condition)

$$\Rightarrow D e^{-\alpha x} + F e^{\alpha x} = D e^{\alpha x} + F e^{-\alpha x}$$

$$D(e^{\alpha x} - e^{-\alpha x}) = F(e^{\alpha x} - e^{-\alpha x})$$

$$(D-F)(e^{\alpha x} - e^{-\alpha x}) = 0$$

b/c $(e^{\alpha x} - e^{-\alpha x})$ cannot be 0 for any $x$

$$\Rightarrow D - F = 0 \qquad D = F$$

You can also check that for odd solution, using $\psi(x) = -\psi(-x)$, one gets $D = -F$.

Continuity of $\psi(x)$ at $x = a$:

$$A e^{-\alpha a} = B\sin(qa) + C\cos(qa) \quad ①$$

Continuity of $d\psi(x)/dx$ at $x=a$

$$-\alpha A e^{-\alpha a} = Bq\cos(qa) - Cq\sin(qa) \quad (2)$$

Continuity of $\psi(x)$ at $x=b$

$$B\sin(qb) + C\cos(qb) = D\left(e^{-\alpha b} + e^{\alpha b}\right) \quad (3)$$

Continuity of $d\psi(x)/dx$ at $x=b$

$$Bq\cos(qb) - Cq\sin(qb) = D\alpha\left(e^{\alpha b} - e^{-\alpha b}\right) \quad (4)$$

(1) $\times(\alpha)$ + (2) :

$$B\alpha\sin(qa) + C\alpha\cos(qa) + Bq\cos(qa) - Cq\sin(qa) = 0$$

$$B[\alpha\sin(qa) + q\cos(qa)] = C[q\sin(qa) - \alpha\cos(qa)]$$

$$C = B\cdot\frac{\alpha\sin(qa) + q\cos(qa)}{q\sin(qa) - \alpha\cos(qa)} \quad (5)$$

(3) $\times(\alpha)$ / (4) :

$$\frac{B\alpha\sin(qb) + C\alpha\cos(qb)}{Bq\cos(qb) - Cq\sin(qb)} = \frac{e^{\alpha b} + e^{-\alpha b}}{e^{\alpha b} - e^{-\alpha b}} = \frac{1}{\tanh(\alpha b)}$$

$$[B\alpha\sin(qb) + C\alpha\cos(qb)]\tanh(\alpha b) = Bq\cos(qb) - Cq\sin(qb)$$

Substitute (5) into the above equation

$$\tanh(\alpha b)\left[\cancel{B}\alpha\sin(qb)+\alpha\cos(qb)\cdot\cancel{B}\cdot\frac{\alpha\sin(qa)+q\cos(qa)}{q\sin(qa)-\alpha\cos(qa)}\right]$$

$$=\cancel{B}q\cos(qb)-q\sin(qb)\cdot\cancel{B}\cdot\frac{\alpha\sin(qa)+q\cos(qa)}{q\sin(qa)-\alpha\cos(qa)}$$

$$\Rightarrow \tanh(\alpha b)\left[\alpha q\sin(qa)\sin(qb)-\alpha^2\sin(qb)\cos(qa)+\alpha^2\sin(qa)\cos(qb)+\alpha q\cos(qa)\cos(qb)\right]$$

$$=q^2\sin(qa)\cos(qb)-\alpha q\cos(qa)\cos(qb)-\alpha q\sin(qa)\sin(qb)-q^2\cos(qa)\sin(qb)$$

$$\Rightarrow \tanh(\alpha b)\left[\alpha q\cos q(a-b)+\alpha^2\sin q(a-b)\right]$$

$$=q^2\sin q(a-b)-\alpha q\cos q(a-b)$$

$$\Rightarrow \tanh(\alpha b)\left[\alpha q+\alpha^2\tan q(a-b)\right]=q^2\tan q(a-b)-\alpha q$$

$$\Rightarrow \tan q(a-b)=\frac{q\alpha\,(1+\tanh\alpha b)}{q^2-\alpha^2\tanh\alpha b}$$

## Odd Solution

The derivation process is quite similar except that $F=-D$. I will not repeat the derivation here.