Problem 1 Griffiths 1.5

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

(a)

$$1 = \int_{-\infty}^{\infty} |\Psi|^2 dx = \int_{-\infty}^{\infty} \Psi\Psi^* dx = A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} dx = 2A^2 \int_0^{\infty} e^{-2\lambda x} dx$$

$$= 2A^2 \left(\frac{e^{-2\lambda x}}{-2\lambda}\right)\Big|_0^{\infty} = \frac{A^2}{\lambda} \quad \Rightarrow \quad A = \sqrt{\lambda}$$

(b)

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx = A^2 \int_{-\infty}^{\infty} xe^{-2\lambda|x|} dx = 0 \ \ (odd\ integrand)$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2|\Psi|^2 dx = A^2 \int_{-\infty}^{\infty} x^2 e^{-2\lambda|x|} dx = 2A^2 \int_0^{\infty} x^2 e^{-2\lambda x} dx$$

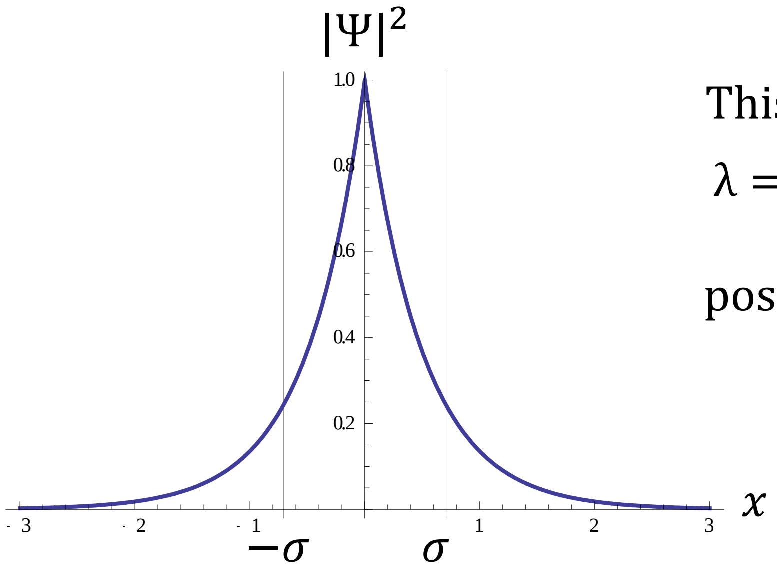
You can do the integration using Mathematica if you don’t want to do it by hand. Type the following line in Mathematica then “Shift + Enter”:

$$\mathrm{Integrate}[x\text{^}2 * \mathrm{Exp}[-2 * \lambda * x], \{x, 0, \mathrm{Infinity}\}]$$

You will get $1/4\lambda^3$

So $$\langle x^2 \rangle = \frac{2A^2}{4\lambda^3} = \frac{2\lambda}{4\lambda^3} = \frac{1}{2\lambda^2}$$

(c) $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda^2} \quad \Rightarrow \quad \textit{standard deviation } \sigma = \frac{1}{\sqrt{2}\lambda}$



This figure plots $|\Psi|^2$ as a function of x for $\lambda = 1$. The two vertical lines shows the positions of $\sigma \left(= \frac{1}{\sqrt{2}} \right)$ and $-\sigma$.

Probability outside the range $(-\sigma, \sigma)$:

$$P_{out} = \int_{-\infty}^{-\sigma} |\Psi|^2 dx + \int_{\sigma}^{\infty} |\Psi|^2 dx = 2A^2 \int_{\sigma}^{\infty} e^{-2\lambda x} dx = \frac{2A^2}{-2\lambda} e^{-2\lambda x} \Big|_{\sigma}^{\infty}$$

$$= e^{-\sqrt{2}} = 0.2431$$

Problem 2

(a) $$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

(b) $$\varphi_0(x) = A \exp\left[\frac{-m\omega x^2}{2\hbar}\right]$$

$$1 = \int_{-\infty}^{\infty} |\varphi_0|^2 dx = A^2 \int_{-\infty}^{\infty} \exp\left[\frac{-m\omega x^2}{\hbar}\right] dx = A^2 \sqrt{\frac{\pi\hbar}{m\omega}} \Rightarrow A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

(c) $$\varphi_1(x) = (B + Cx)\exp\left[\frac{-m\omega x^2}{2\hbar}\right]$$

Orthogonality condition:

$$0 = \langle \varphi_0 | \varphi_1 \rangle = \int_{-\infty}^{\infty} \varphi_0{}^* \varphi_1 dx = \int_{-\infty}^{\infty} A(B + Cx)\exp\left[\frac{-m\omega x^2}{\hbar}\right] dx$$

$$= AB \int_{-\infty}^{\infty} \exp\left[\frac{-m\omega x^2}{\hbar}\right] dx + \underbrace{\cancel{AC \int_{-\infty}^{\infty} x\exp\left[\frac{-m\omega x^2}{\hbar}\right] dx}}_{= 0}$$

$$= AB \sqrt{\frac{\pi\hbar}{m\omega}} \Rightarrow B = 0$$

Normalization condition:

$$1 = \langle \varphi_1 | \varphi_1 \rangle = \int_{-\infty}^{\infty} |\varphi_1|^2 dx = C^2 \int_{-\infty}^{\infty} x^2 \exp\left[\frac{-m\omega x^2}{\hbar}\right] dx = C^2 \frac{\sqrt{\pi}}{2}\left(\frac{\hbar}{m\omega}\right)^{3/2}$$

$$\Rightarrow \quad C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}}$$

$$\varphi_2(x) = (D + Ex + Fx^2)\exp\left[\frac{-m\omega x^2}{2\hbar}\right]$$

Orthogonality conditions: $\langle \varphi_0 | \varphi_2 \rangle = 0 \quad \& \quad \langle \varphi_1 | \varphi_2 \rangle = 0$

$$0 = \langle \varphi_0 | \varphi_2 \rangle = \int_{-\infty}^{\infty} {\varphi_0}^* \varphi_2 dx = \int_{-\infty}^{\infty} A(D + Ex + Fx^2)\exp\left[\frac{-m\omega x^2}{\hbar}\right] dx$$

= 0

$$= AD\int_{-\infty}^{\infty} \exp\left[\frac{-m\omega x^2}{\hbar}\right] dx + \cancel{AE\int_{-\infty}^{\infty} x\exp\left[\frac{-m\omega x^2}{\hbar}\right] dx} + AF\int_{-\infty}^{\infty} x^2\exp\left[\frac{-m\omega x^2}{\hbar}\right] dx$$

$$= AD\sqrt{\frac{\pi\hbar}{m\omega}} + AF\frac{\sqrt{\pi}}{2}\left(\frac{\hbar}{m\omega}\right)^{3/2} \quad \Rightarrow \quad F = -2D\left(\frac{m\omega}{\hbar}\right)$$

$$0 = \langle \varphi_1 | \varphi_2 \rangle = \int_{-\infty}^{\infty} \varphi_1{}^* \varphi_2 dx = \int_{-\infty}^{\infty} Cx(D + Ex + Fx^2)\exp\left[\frac{-m\omega x^2}{\hbar}\right] dx$$

$$= \underbrace{\cancel{CD \int_{-\infty}^{\infty} x\exp\left[\frac{-m\omega x^2}{\hbar}\right] dx}}_{= 0} + CE \int_{-\infty}^{\infty} x^2\exp\left[\frac{-m\omega x^2}{\hbar}\right] dx + \underbrace{\cancel{CF \int_{-\infty}^{\infty} x^3\exp\left[\frac{-m\omega x^2}{\hbar}\right] dx}}_{= 0}$$

$$= CE \frac{\sqrt{\pi}}{2}\left(\frac{\hbar}{m\omega}\right)^{3/2} \quad \Rightarrow \quad E = 0$$

Normalization condition:

$$1 = \langle \varphi_2 | \varphi_2 \rangle = \int_{-\infty}^{\infty} |\varphi_2|^2 dx = \int_{-\infty}^{\infty} (D + Fx^2)^2\exp\left[\frac{-m\omega x^2}{\hbar}\right] dx$$

$$= D^2 \int_{-\infty}^{\infty} \exp\left[\frac{-m\omega x^2}{\hbar}\right] dx + 2DF \int_{-\infty}^{\infty} x^2\exp\left[\frac{-m\omega x^2}{\hbar}\right] dx + F^2 \int_{-\infty}^{\infty} x^4\exp\left[\frac{-m\omega x^2}{\hbar}\right] dx$$

$$= D^2 \sqrt{\frac{\pi\hbar}{m\omega}} + 2DF \frac{\sqrt{\pi}}{2}\left(\frac{\hbar}{m\omega}\right)^{3/2} + F^2 \frac{3\sqrt{\pi}}{4}\left(\frac{\hbar}{m\omega}\right)^{5/2} \qquad Recall\ \ F = -2D\left(\frac{m\omega}{\hbar}\right)$$

$$= 2D^2 \sqrt{\frac{\pi\hbar}{m\omega}} \quad \Longrightarrow \quad D = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \quad \text{and} \quad F = -\frac{\sqrt{2}}{\pi^{1/4}}\left(\frac{m\omega}{\hbar}\right)^{5/4}$$

(d) $$\varphi_1(x) = Cx\exp\left[\frac{-m\omega x^2}{2\hbar}\right] \qquad \text{with } C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}$$

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2$$

$$\begin{aligned}
H\varphi_1(x) &= \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2\right)Cx\exp\left[\frac{-m\omega x^2}{2\hbar}\right] \\
&= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left\{Cx\exp\left[\frac{-m\omega x^2}{2\hbar}\right]\right\} + \frac{1}{2}m\omega^2x^2Cx\exp\left[\frac{-m\omega x^2}{2\hbar}\right] \\
&= -\frac{\hbar^2}{2m}\left\{-\frac{3m\omega}{\hbar}x + \frac{m^2\omega^2}{\hbar^2}x^3\right\}C\exp\left[\frac{-m\omega x^2}{2\hbar}\right] \\
&\quad + \frac{1}{2}m\omega^2x^3C\exp\left[\frac{-m\omega x^2}{2\hbar}\right] \\
&= \frac{3\hbar\omega}{2}Cx\exp\left[\frac{-m\omega x^2}{2\hbar}\right] \\
&= \frac{3\hbar\omega}{2}\varphi_1(x) \quad \Longrightarrow \quad E_1 = \frac{3\hbar\omega}{2}
\end{aligned}$$

Problem 3

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x) \qquad\qquad a_- = \frac{1}{\sqrt{2\hbar m\omega}}(+ip + m\omega x)$$

$$[a_-, a_+] = \frac{1}{2\hbar m\omega}[ip + m\omega x, -ip + m\omega x] = \frac{1}{2\hbar m\omega}\{im\omega[p, x] - im\omega[x, p]\}$$

$$= \frac{1}{2\hbar m\omega}\{im\omega(-i\hbar) - im\omega(i\hbar)\} = 1$$

$$\hbar\omega\left(a_+ a_- + \frac{1}{2}\right) = \hbar\omega\left[\frac{1}{2\hbar m\omega}(-ip + m\omega x)(ip + m\omega x) + \frac{1}{2}\right]$$

$$= \frac{1}{2m}(p^2 + im\omega xp - im\omega px + m^2\omega^2 x^2) + \frac{1}{2}\hbar\omega$$

$$= \frac{1}{2m}(p^2 + im\omega[x, p] + m^2\omega^2 x^2) + \frac{1}{2}\hbar\omega$$

$$= \frac{1}{2m}(p^2 - \hbar m\omega + m^2\omega^2 x^2) + \frac{1}{2}\hbar\omega$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \quad = \quad H$$

$$H(a_+\varphi_n) = \hbar\omega\left(a_+a_- + \frac{1}{2}\right)a_+\varphi_n = \hbar\omega(a_+a_-a_+ + 1/2a_+)\,\varphi_n$$

$$= \hbar\omega a_+\left(a_-a_+ + \frac{1}{2}\right)\varphi_n = \hbar\omega a_+\left(a_+a_- + 1 + \frac{1}{2}\right)\varphi_n$$

$$= a_+(H + \hbar\omega)\varphi_n = a_+(E + \hbar\omega)\varphi_n = (E + \hbar\omega)a_+\varphi_n$$

Therefore $a_+\varphi_n$ is an eigenstate of H with energy $(E + \hbar\omega)$. But it may not be normalized. It is related to the normalized eigenstate φ_{n+1} up to a coefficient:

$$a_+\varphi_n = C\varphi_{n+1}$$

$$1 = \langle\varphi_{n+1}|\varphi_{n+1}\rangle = \frac{1}{|C|^2}\langle a_+\varphi_n|a_+\varphi_n\rangle = \frac{1}{|C|^2}\langle\varphi_n|a_-a_+\varphi_n\rangle$$

$$= \frac{1}{|C|^2}\langle\varphi_n|(a_+a_- + 1)\varphi_n\rangle = \frac{1}{|C|^2}\langle\varphi_n|(H/\hbar\omega + 1/2)\varphi_n\rangle = \frac{1}{|C|^2}(n + 1)$$

$$\Longrightarrow\quad C = \sqrt{n+1} \quad \text{and} \quad a_+\varphi_n = \sqrt{n+1}\,\varphi_{n+1} \qquad \text{Similarly} \quad a_-\varphi_n = \sqrt{n}\,\varphi_{n-1}$$

$$H\varphi_n = \hbar\omega\left(a_+a_- + \frac{1}{2}\right)\varphi_n = \hbar\omega\left(a_+a_-\varphi_n + \frac{1}{2}\varphi_n\right) = \hbar\omega\left(a_+\sqrt{n}\,\varphi_{n-1} + \frac{1}{2}\varphi_n\right)$$

$$= \hbar\omega\left(\sqrt{n}\sqrt{n}\varphi_n + \frac{1}{2}\varphi_n\right) = \hbar\omega\left(n + \frac{1}{2}\right)\varphi_n$$

Problem 4

Ground state: $\varphi_0(x) = A\exp\left[\frac{-m\omega x^2}{2\hbar}\right]$ with $A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

$$\langle x\rangle = \int_{-\infty}^{\infty}\varphi_0{}^*(x)x\varphi_0(x)dx = A^2\int_{-\infty}^{\infty}x\exp\left[\frac{-m\omega x^2}{\hbar}\right]dx = 0$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty}\varphi_0{}^*(x)x^2\varphi_0(x)dxdx = A^2\int_{-\infty}^{\infty}x^2\exp\left[\frac{-m\omega x^2}{\hbar}\right]dx = A^2\frac{\sqrt{\pi}}{2}\left(\frac{\hbar}{m\omega}\right)^{3/2} = \frac{\hbar}{2m\omega}$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\langle p\rangle = \int_{-\infty}^{\infty}\varphi_0{}^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\varphi_0(x)dx = A^2\int_{-\infty}^{\infty}\exp\left[\frac{-m\omega x^2}{2\hbar}\right](im\omega)x\exp\left[\frac{-m\omega x^2}{2\hbar}\right]dx$$

$$= im\omega A^2\int_{-\infty}^{\infty}x\exp\left[\frac{-m\omega x^2}{\hbar}\right]dx = 0$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty}\varphi_0{}^*(x)\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\varphi_0(x)dx$$

$$= -A^2\ \hbar^2 \int_{-\infty}^{\infty} \exp\left[\frac{-m\omega x^2}{2\hbar}\right]\left(\frac{m^2\omega^2}{\hbar^2}x^2 - \frac{m\omega}{\hbar}\right)\exp\left[\frac{-m\omega x^2}{2\hbar}\right]dx$$

$$= -A^2\ m^2\omega^2 \int_{-\infty}^{\infty} x^2 \exp\left[\frac{-m\omega x^2}{\hbar}\right]dx + A^2\hbar m\omega \int_{-\infty}^{\infty} \exp\left[\frac{-m\omega x^2}{\hbar}\right]dx$$

$$= -A^2\ m^2\omega^2 \frac{\sqrt{\pi}}{2}\left(\frac{\hbar}{m\omega}\right)^{3/2} + A^2\hbar m\omega\sqrt{\frac{\pi\hbar}{m\omega}} = \frac{1}{2}\hbar m\omega$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{1}{2}\hbar m\omega}$$

$$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{1}{2}\hbar m\omega} = \frac{\hbar}{2}$$

Minimum uncertainty

1st excited state:

$$\varphi_1(x) = Cx\exp\left[\frac{-m\omega x^2}{2\hbar}\right] \quad \text{with} \quad C = \frac{\sqrt{2}}{\pi^{1/4}}\left(\frac{m\omega}{\hbar}\right)^{3/4}$$

$$\langle x\rangle = \int_{-\infty}^{\infty} \varphi_1{}^*(x)x\varphi_1(x)dx = C^2\int_{-\infty}^{\infty} x^3\exp\left[\frac{-m\omega x^2}{\hbar}\right]dx = 0$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} \varphi_1{}^*(x)x^2\varphi_1(x)dx = C^2\int_{-\infty}^{\infty} x^4\exp\left[\frac{-m\omega x^2}{\hbar}\right]dx = C^2\frac{3\sqrt{\pi}}{4}\left(\frac{\hbar}{m\omega}\right)^{5/2} = \frac{3\hbar}{2m\omega}$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{3\hbar}{2m\omega}}$$

$$\langle p\rangle = \int_{-\infty}^{\infty} \varphi_1{}^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\varphi_1(x)dx$$

$$= -i\hbar C^2\int_{-\infty}^{\infty} x\left(1 - \frac{m\omega}{\hbar}x^2\right)\exp\left[\frac{-m\omega x^2}{\hbar}\right]dx$$

$$= 0$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \varphi_1{}^*(x) \left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right) \varphi_1(x) dx$$

$$= -C^2\ \hbar^2 \int_{-\infty}^{\infty} \left(\frac{m^2\omega^2}{\hbar^2} x^4 - \frac{3m\omega}{\hbar} x^2\right) \exp\left[\frac{-m\omega x^2}{\hbar}\right] dx$$

$$= -C^2\ m^2\omega^2 \int_{-\infty}^{\infty} x^4 \exp\left[\frac{-m\omega x^2}{\hbar}\right] dx + 3C^2\hbar m\omega \int_{-\infty}^{\infty} x^2 \exp\left[\frac{-m\omega x^2}{\hbar}\right] dx$$

$$= -\frac{3}{2}\hbar m\omega + 3\hbar m\omega = \frac{3}{2}\hbar m\omega$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{3}{2}\hbar m\omega}$$

$$\sigma_x \sigma_p = \sqrt{\frac{3\hbar}{2m\omega}} \sqrt{\frac{3}{2}\hbar m\omega} = \frac{3\hbar}{2} > \frac{\hbar}{2}$$

Problem 4: operator approach: MUCH simpler!

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \qquad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

$$\langle n|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|a_+ + a_-|n\rangle = 0$$

$$\langle n|x^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|a_+a_+ + a_+a_- + a_-a_+ + a_-a_-|n\rangle = \frac{\hbar}{m\omega}(n + 1/2)$$

$$\langle n|p|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle n|a_+ - a_-|n\rangle = 0$$

$$\langle n|p^2|n\rangle = -\frac{\hbar m\omega}{2}\langle n|a_+a_+ - a_+a_- - a_-a_+ + a_-a_-|n\rangle = \hbar m\omega(n + 1/2)$$

Ground state: $$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{1}{2}\hbar m\omega} = \frac{\hbar}{2}$$

1st excited state: $$\sigma_x\sigma_p = \sqrt{\frac{3\hbar}{2m\omega}}\sqrt{\frac{3}{2}\hbar m\omega} = \frac{3\hbar}{2}$$

Problem 5

(a) Probability to be in the ground state =

$$|\langle\varphi_1(x)|\psi_A(x,t=0)\rangle|^2 = \left|\left\langle\varphi_1(x)\middle|\frac{1}{\sqrt{2}}\varphi_1(x)+\frac{1}{\sqrt{2}}\varphi_2(x)\right\rangle\right|^2 = \frac{1}{2}$$

(b)
$$\psi_A(x,t) = \frac{1}{\sqrt{2}}\varphi_1(x)e^{-iE_1t/\hbar}+\frac{1}{\sqrt{2}}\varphi_2(x)e^{-iE_2t/\hbar}$$

For a 1D infinite square well $E_n = \dfrac{n^2\pi^2\hbar^2}{2ma^2}$, $E_1 = \dfrac{\pi^2\hbar^2}{2ma^2}$, $E_2 = \dfrac{2\pi^2\hbar^2}{ma^2}$,

$$\psi_A(x,t) = \frac{1}{\sqrt{2}}\varphi_1(x)\exp\left(-\frac{i\pi^2\hbar t}{2ma^2}\right)+\frac{1}{\sqrt{2}}\varphi_2(x)\exp\left(-\frac{i2\pi^2\hbar t}{ma^2}\right)$$

(c) Density distribution at $t=0$ is:

$$|\psi_A(x,t=0)|^2 = {\psi_A}^*(x,t=0)\psi_A(x,t=0)$$

$$= \left(\frac{1}{\sqrt{2}}{\varphi_1}^*(x)+\frac{1}{\sqrt{2}}{\varphi_2}^*(x)\right)\left(\frac{1}{\sqrt{2}}\varphi_1(x)+\frac{1}{\sqrt{2}}\varphi_2(x)\right)$$

$$= \frac{1}{2}(|\varphi_1|^2 + |\varphi_2|^2 + 2\varphi_1\varphi_2)$$ since φ_1 & φ_2 are real

$$|\psi_A(x,t)|^2 = {\psi_A}^*(x,t)\psi_A(x,t)$$

$$= \left(\frac{1}{\sqrt{2}}{\varphi_1}^*(x)e^{iE_1t/\hbar} + \frac{1}{\sqrt{2}}{\varphi_2}^*(x)e^{iE_2t/\hbar}\right)\left(\frac{1}{\sqrt{2}}\varphi_1(x)e^{-iE_1t/\hbar} + \frac{1}{\sqrt{2}}\varphi_2(x)e^{-iE_2t/\hbar}\right)$$

$$= \frac{1}{2}\left(|\varphi_1|^2 + |\varphi_2|^2 + {\varphi_1}^*\varphi_2 e^{-i(E_2-E_1)t/\hbar} + \varphi_1{\varphi_2}^* e^{i(E_2-E_1)t/\hbar}\right)$$

$$= \frac{1}{2}(|\varphi_1|^2 + |\varphi_2|^2 + 2\varphi_1\varphi_2\cos((E_2-E_1)t/\hbar))$$

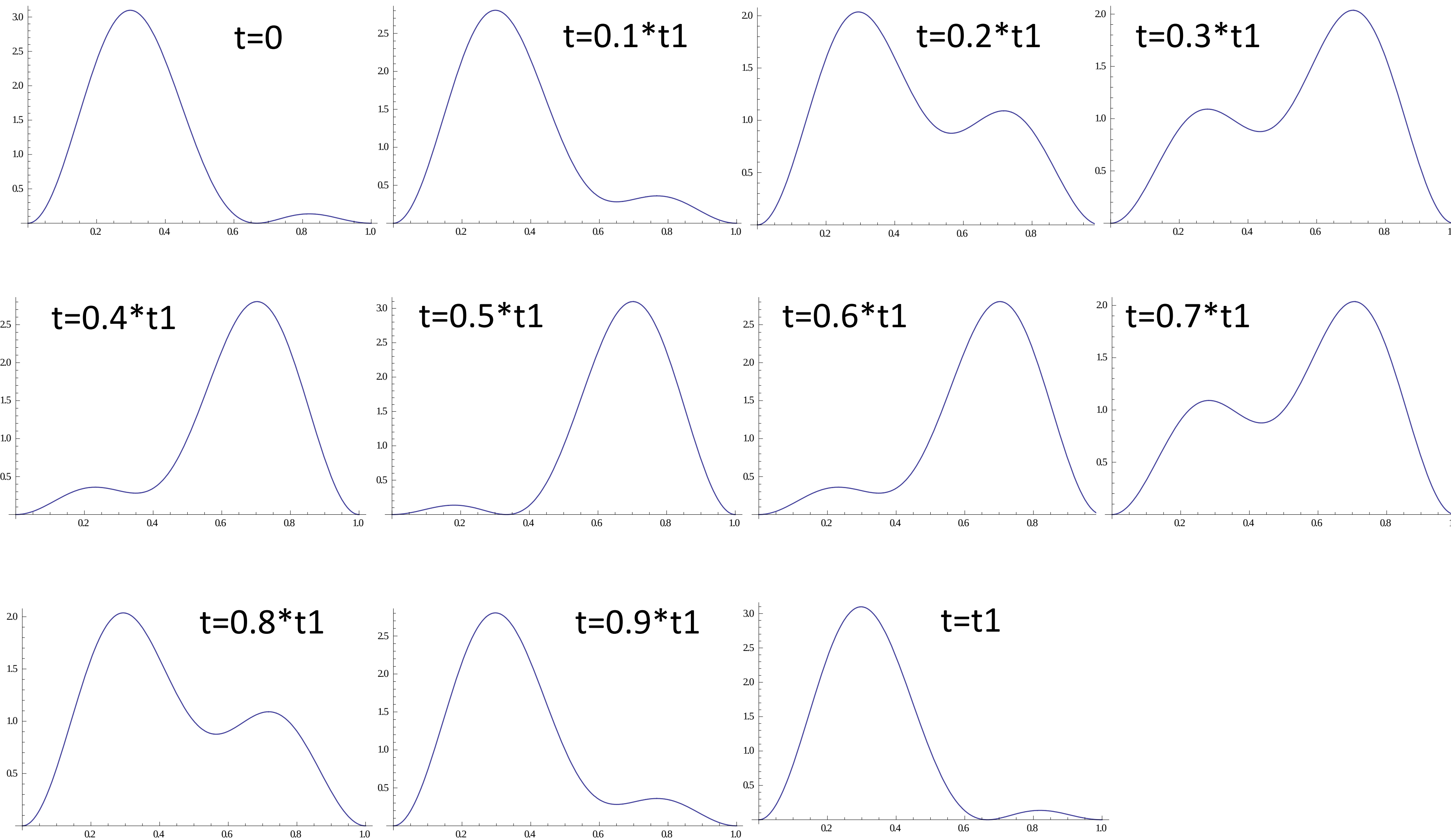
Therefore $|\psi_A(x,t)|^2 = |\psi_A(x,t=0)|^2$ when $\cos((E_2-E_1)t/\hbar) = 1$

Or $(E_2-E_1)t/\hbar = 2\pi, 4\pi, 6\pi, \ldots$

$$t = 2n\pi\hbar/(E_2-E_1) = 2n\pi\hbar/\left[\frac{2\pi^2\hbar^2}{ma^2} - \frac{\pi^2\hbar^2}{2ma^2}\right] = \frac{4nma^2}{3\pi\hbar}$$ where $n = 1,2,3,\ldots$

The smallest $t_1 = \dfrac{4ma^2}{3\pi\hbar}$

(d)


t=0
t=0.1*t1
t=0.2*t1
t=0.3*t1
t=0.4*t1
t=0.5*t1
t=0.6*t1
t=0.7*t1
t=0.8*t1
t=0.9*t1
t=t1

(e)

$$\psi_B(x,t) = \frac{1}{\sqrt{2}}\varphi_1(x)e^{-iE_1t/\hbar} - \frac{1}{\sqrt{2}}\varphi_2(x)e^{-iE_2t/\hbar}$$

Density distribution at $t = 0$ is:

$$|\psi_B(x,t=0)|^2 = \psi_B{}^*(x,t=0)\psi_B(x,t=0)$$

$$= \left(\frac{1}{\sqrt{2}}\varphi_1{}^*(x) - \frac{1}{\sqrt{2}}\varphi_2{}^*(x)\right)\left(\frac{1}{\sqrt{2}}\varphi_1(x) - \frac{1}{\sqrt{2}}\varphi_2(x)\right)$$

$$= \frac{1}{2}(|\varphi_1|^2 + |\varphi_2|^2 - 2\varphi_1\varphi_2) \qquad \text{since } \ \varphi_1 \,\&\, \varphi_2 \text{ are real}$$

$$|\psi_B(x,t)|^2 = \psi_B{}^*(x,t)\psi_B(x,t)$$

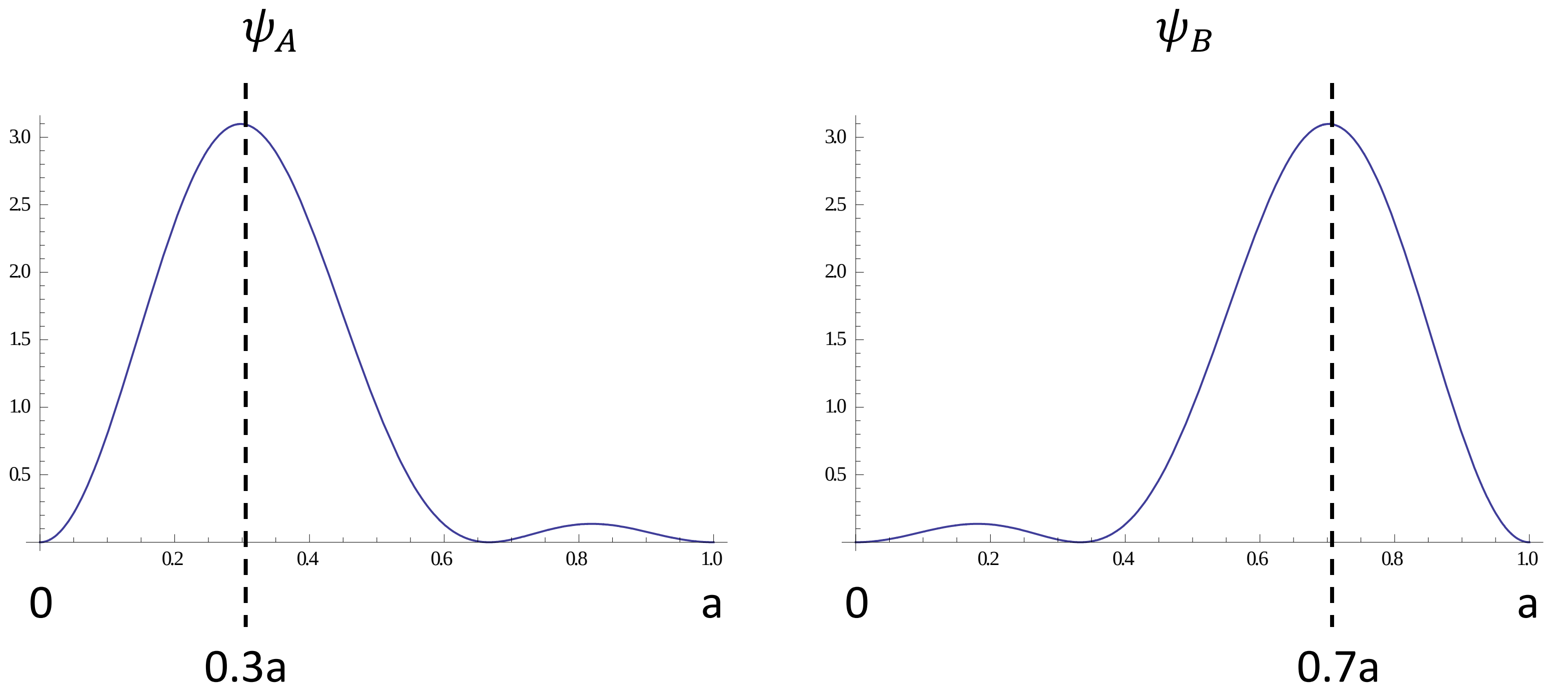
$$= \left(\frac{1}{\sqrt{2}}\varphi_1{}^*(x)e^{iE_1t/\hbar} - \frac{1}{\sqrt{2}}\varphi_2{}^*(x)e^{iE_2t/\hbar}\right)\left(\frac{1}{\sqrt{2}}\varphi_1(x)e^{-iE_1t/\hbar} - \frac{1}{\sqrt{2}}\varphi_2(x)e^{-iE_2t/\hbar}\right)$$

$$= \frac{1}{2}\left(|\varphi_1|^2 + |\varphi_2|^2 - \varphi_1{}^*\varphi_2 e^{-i(E_2-E_1)t/\hbar} - \varphi_1\varphi_2{}^* e^{i(E_2-E_1)t/\hbar}\right)$$

$$= \frac{1}{2}(|\varphi_1|^2 + |\varphi_2|^2 - 2\varphi_1\varphi_2\cos((E_2-E_1)t/\hbar))$$

t_1 *is the same as the previous case*

(f)



The two initial states can be determined by measuring the position of the particle:

If the measured value of x is around 0.3a then the initial state is ψ_A;

If the measured value of x is around 0.7a then the initial state is ψ_B.