Summary of potential Scattering Theory A. Scattering by a Short-Range Potential

(i) Plane Wave

$$\begin{aligned}\Psi^+_{\mathbf{k}}(\mathbf{r}) &= e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikz} \\ \Psi^-_{\mathbf{k}}(\mathbf{r}) &= e^{-i\mathbf{k}\cdot\mathbf{r}}\end{aligned}$$

(ii) Partial Wave Expansion

$$\Psi^+_{\mathbf{k}}(\mathbf{r}) = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)$$

(iii) Short-Range Potential is for $V(r) \xrightarrow{r\to\infty} 0$ faster than $1/r^2$
Two independent solutions in the region $V(r) = 0$

$$j_l(kr) \quad \text{and} \quad n_l(kr)$$

$$\begin{aligned}j_l(kr) &\xrightarrow{r\to\infty} \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2}\right) \\ n_l(kr) &\xrightarrow{r\to\infty} -\frac{1}{kr}\cos\left(kr - \frac{l\pi}{2}\right)\end{aligned}$$

(iv) Scattering Wave with momentum $\mathbf{k}$ in a Short-Range Potential

$$\Psi_{\mathbf{k}}(\mathbf{r}) = \sum_{l=0}^{\infty}(2l+1)i^l f_l(kr)P_l(\cos\theta)e^{i\delta_l}$$

$$\begin{aligned}&f_l(kr) \xrightarrow{r\to\infty} \frac{1}{kr}\left[\sin\left(kr - \frac{l\pi}{2}\right) + K_l\cos\left(kr - \frac{l\pi}{2}\right)\right] \\ &\Rightarrow \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2} + \delta_l\right)\end{aligned}$$

$$K_l = \tan\delta_l$$

$$\Psi_{\mathbf{k}}(\mathbf{r}) \xrightarrow{r\to\infty} e^{ikz} + f_{\mathbf{k}}(\theta)\frac{e^{ikr}}{r}$$

$$f_{\mathbf{k}}(\theta) = \frac{1}{2ik}\sum_l (2l+1)(e^{2i\delta_l} - 1)P_l(\cos\theta)$$

$$\frac{d\sigma}{d\Omega} = |f_{\mathbf{k}}(\theta)|^2$$

$$\sigma_{\text{Tot}} = \sum_l \sigma_l = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2\delta_l$$

(v) K, T and S “matrices”

$$T_l = \tan\delta_l$$

$$K_l = \frac{T_l}{1 + iT_l}$$

$$S_l = 1 + 2iT_l = e^{2i\delta_l}$$

Note: different partial waves will interfere in the differential cross sections. They do not interfere for the total cross sections.

B. Scattering by a Long-Range Potential

(i) Coulomb Waves can be found in textbooks (Schiff p.138-). Let $\xi = r - z$, $\eta = r + z$, then the Coulomb function with fixed momentum $\mathbf{k}$ is given by

$$\Psi_{\mathbf{k}}^{C+} = e^{ikz} f(\xi) = Ce^{kz} F(-in, 1, ik\xi)$$

$$n = ZZ'e^2/\hbar\breve{}$$

where the two charges are given by Z and Z', and v is the velocity,

$$F(a, b, z) = 1 + \frac{az}{b} + \frac{a(a+1)z^2}{b(b+1)2!} + ...$$

is the confluent hypergeometric function.

Asymptotic form of the Coulomb function (Schiff eq. (21.a))

$$\Psi_{\mathbf{k}}^{C+} \xrightarrow{r\to\infty} C\frac{e^{n\pi/2}}{\Gamma(1+in)}\left\{e^{ikz+n\ln(r-z)k}\left[1 + \frac{n^2}{ik(r-z)}\right] + \frac{e^{i(kr-n\ln 2kr)}}{r} f_c(\theta)\right\}$$

It is expressed as an incident wave and a scattered wave. The Coulomb scattering amplitude is

$$f_c(\theta) = \frac{n}{2k\sin^2\theta/2} e^{in\ln(\sin^{-2}\frac{\theta}{2}) + i\pi + 2i\eta_0}$$

$$\eta_0 = \arg\Gamma(1 + in)$$

which gives the same differential cross sections as the Rutherford scattering cross section derived classically. The C depends on the normalization convention.

(ii) Partial Wave Expansion of Coulomb functions

$$\Psi_{\mathbf{k}}^{+}(\mathbf{r}) = \sum_{l=0}^{\infty}(2l+1)i^l e^{i\eta_l} F_l(kr) P_l(\cos\theta)$$

$$F_l(kr) \xrightarrow{r\to\infty} \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2} - n\ln 2kr + \eta_l\right)$$

$$\eta_l = \arg\Gamma(l + 1 + in)$$

(iii) Regular and Irregular Coulomb Wave Functions $F_l(kr)$ and $G_l(kr)$ where the irregular Coulomb function is

$$G_l(kr) \xrightarrow{r\to\infty} -\frac{1}{kr}\cos\left(kr - \frac{l\pi}{2} - n\ln 2kr + \eta_l\right)$$

Both are solutions of Schrödinger radial equation for a Coulomb potential.

(iv) Scattering Wave in a Modified Coulomb Potential

$$V(r) = V_c(r) + V_s(r)$$

$$\Psi^+_{\mathbf{k}}(\mathbf{r}) = \sum_{l=0}^{\infty}(2l+1)i^l e^{i(\eta_l+\delta_l)} f_l(kr) P_l(\cos\theta)$$

$$\left[\Psi^+_{\mathbf{k}}(\mathbf{r}) - \Psi^{C+}_{\mathbf{k}}(\mathbf{r})\right] \xrightarrow{r\to\infty} f_s(\theta)\frac{1}{r}e^{i(kr-n\ln 2kr)}$$

Thus the asymptotic form of $\Psi^+_{\mathbf{k}}$ is

$$\Psi^+_{\mathbf{k}}(\mathbf{r}) \xrightarrow{r\to\infty} e^{i[kz+n\ln k(r-z)]} + [f_c(\theta) + f_s(\theta)]\frac{1}{r}e^{i(kr-n\ln 2kr)}$$

$$\frac{d\sigma}{d\Omega} = |f_c(\theta) + f_s(\theta)|^2$$

$$f_s(\theta) = \sum_{l=0}^{\infty}\frac{1}{2ik}(2l+1)e^{2i\eta_l}(e^{2i\delta_l} - 1)P_l(\cos\theta)$$

Note that sum over l can be truncated since δ_l involves the short-range potential only.

(v) Scattering Waves Describing Ionization (Incoming Scattering Waves)

$$\Psi^-_{\mathbf{k}}(\mathbf{r}) = \sum_{l=0}^{\infty}(2l+1)i^l e^{-i(\eta_l+\delta_l)} f_l(kr) P_l(\cos\theta)$$

Recall

$$P_l(\cos\theta) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\hat{k})Y_{lm}(\hat{r})$$