Many-body wave scattering by small bodies and creating materials with a desired refraction coefficient

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Abstract

Many-body scattering problem is solved asymptotically when the size of the particles tends to zero and the number of the particles tends to infinity.

A method is given for calculation of the number of small particles and their boundary impedances such that embedding of these particles in a bounded domain, filled with known material, results in creating a new material with a desired refraction coefficient.

The new material may be created so that it has negative refraction, that is, the group velocity in this material is directed opposite to the phase velocity.

Another possible application consists of creating the new material with some desired wave-focusing properies. For example, one can create a new material which scatters plane wave mostly in a fixed given solid angle. In this application it is assumed that the incident plane wave has a fixed frequency and a fixed incident direction.

An inverse scattering problem with scattering data given at a fixed wave number and at a fixed incident direction is formulated and solved.

Acoustic and electromagnetic (EM) wave scattering problems are discussed.

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Wave scattering by many small particles and creating materials with a desired refraction coefficient

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Scattering problem

$$L_0 u_0 := [\nabla^2 + k^2 n_0^2(x)] u_0 := [\nabla^2 + k^2 - q_0(x)] u_0 = 0 \text{ in } \mathbb{R}^3$$

$$u_0 = e^{ik\alpha\cdot x} + v_0, \quad \lim_{r\to\infty} r\left(\frac{\partial v_0}{\partial r} - ikv_0\right) = 0, \quad r := |x|,$$
$$\Im n_0^2(x) \ge 0, \quad \alpha \in S^2, \quad k = \text{const} > 0.$$

$$L_0 G = -\delta(x-y) \text{ in } \mathbb{R}^3; \qquad n_0^2(x) = 1 - k^{-2} q_0(x)$$
$$q_0(x) = k^2 - k^2\, n_0^2(x).$$

Many-body scattering problem

$$\begin{cases} L_0 u_M = 0 \text{ in } \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m; & D_m = B_m(x_m, a) \\ \frac{\partial u_M}{\partial N} = \zeta_m u_M \text{ on } S_m := \partial D_m, & \zeta_m = \frac{h(x_m)}{a^\kappa}, \quad 0 < \kappa \le 1, \\ u_M = u_0 + v_M, \end{cases}$$

where N is the outer unit normal to S_m, and $h(x) \in C(D)$ is an arbitrary function, $h = h_1 + ih_2$, $h_2 \le 0$. Let $d := \min_{m \neq j} \operatorname{dist}(x_m, x_j)$. We assume that:

$$ka \ll 1, \quad d \gg a, \quad d = O(a^{(2-\kappa)/3}),$$

$$\mathcal{N}(\Delta) := \sum_{x_m \subset \Delta} 1 = \frac{1}{a^{2-\kappa}} \int_\Delta N(x) dx [1 + o(1)], \quad a \to 0.$$

$$M = M(a) \sim O(a^{-(2-\kappa)}), \qquad 0 < \kappa < 1.$$

$$
\begin{aligned}
u_M(x) &= u_0(x) + \sum_{m=1}^{M} \int_{S_m} G(x,t)\sigma_m(t)dt \\
&= u_0(x) + \sum_{m=1}^{M} G(x,x_m)Q_m + \sum_{m=1}^{M} J_m.
\end{aligned}
$$

$$
Q_m := \int_{S_m} \sigma_m(t)dt, \quad J_m := \int_{S_m} [G(x,t) - G(x,x_m)]\sigma_m(t)dt,
$$

$$
I_m := |G(x,x_m)Q_m|.
$$

$$
|J_m| \ll I_m \qquad |x - x_m| \gg a.
$$

If $|J_m| \ll I_m, \quad a \to 0$, then

$$u_M(x) \simeq u_0(x) + \sum_{m=1}^{M} G(x, x_m) Q_m, \quad |x - x_m| \geq d \gg a.$$

Define

$$u_e := u_e^{(m)} := u_M(x) - \int_{S_m} G(x,t)\sigma_m(t)dt, \quad |x - x_m| \sim a.$$

If $|x - x_m| \gg a$, then $u_e \sim u_M$ as $a \to 0$.

We prove below that

$$Q_m \sim -4\pi u_e(x_m)\, h(x_m) a^{2-\kappa}.$$

The $u_e(x_m)$ are found from linear algebraic system:

$$u_e(x_m) = u_0(x_m) - 4\pi a^{2-\kappa} \sum_{m' \neq m}^{M} G(x_m, x_{m'}) u_e(x_{m'})\, h(x_{m'}).$$

Formula for Q_m

$$u_{e_N} - \zeta_m u_e + \frac{A_m\sigma_m - \sigma_m}{2} - \zeta_m T_m \sigma_m = 0 \quad \text{on } S_m.$$

$$A_m\sigma_m := 2\int_{S_m} \frac{\partial G(s,t)}{\partial N_s}\sigma_m(t)dt, \quad T_m\sigma_m := \int_{S_m} G(s,t)\sigma_m(t)dt.$$

$$G(x,y) = \frac{1}{4\pi|x-y|}[1 + O(|x-y|)], \quad |x-y| \to 0.$$

$$\frac{4}{3}\pi a^3 \Delta u_e(x_m) - \zeta_m 4\pi a^2 u_e(x_m) = Q_m + \zeta_m \int_{S_m} ds \int_{S_m} \frac{\sigma_m(t)dt}{4\pi|s-t|},$$

$$\int_{S_m} A\sigma_m dt = -\int_{S_m} \sigma_m dt,$$

$$\int_{S_m} \frac{ds}{4\pi|s-t|} = a$$

$$\frac{4}{3}\pi a^3 \Delta u_e(x_m) - 4\pi\zeta_m u_e(x_m)a^2 = Q_m(1+\zeta_m a).$$

$$Q_m = \frac{a^3[\frac{4\pi}{3}\Delta u_e(x_m) - 4\pi u_e(x_m)\zeta_m a^{-1}]}{1+\zeta_m a}.$$

If $\zeta_m = \frac{h(x_m)}{a^\kappa}$, $\kappa < 1$, then

$$\boxed{Q_m \sim -4\pi u_e(x_m)\, h(x_m) a^{2-\kappa}.}$$

If $\kappa = 1$, then

$$\boxed{Q_m \sim -4\pi u_e(x_m)\, \frac{h(x_m)}{1+h(x_m)}\, a.}$$

Formula for σ_m

$$u_M = u_e + \sigma_m \int_{S_m} \frac{dt}{4\pi|x-t|} = u_e + \frac{\sigma_m a^2}{|x|}, \ \ |x| = O(a).$$

$$u_{e_N} - \frac{h(x_m)}{a^\kappa} u_e - \sigma_m - \frac{h(x_m)}{a^\kappa} \sigma_m a = 0$$

$$\sigma_m = \frac{u_{e_N} - h(x_m) u_e(x_m) a^{-\kappa}}{1 + h(x_m) a^{1-\kappa}}$$

If $\kappa < 1$, $a \to 0$, then $\boxed{\sigma_m \sim -h(x_m) u_e(x_m) a^{-\kappa}.}$

If $\kappa = 1$, $a \to 0$, then $\boxed{\sigma_m \sim -\frac{h(x_m)}{1 + h(x_m)} u_e(x_m) a^{-1}.}$

If $1 < \kappa < 2$, $a \to 0$, then $\boxed{\sigma_m \sim -\frac{u_e(x_m)}{a}.}$

When is $I_m \gg |J_m|$?

$|G(x, x_m)Q_m| = I_m \sim \frac{a^{2-\kappa}}{a^{(2-\kappa)/3}}, \quad J_m \sim \frac{a}{a^{2(2-\kappa)/3}} \cdot a^{2-\kappa} \sim a^{3-\kappa-2(2-\kappa)/3}.$

$$\boxed{a^{2-\kappa-(2-\kappa)/3} \gg a^{3-\kappa-2(2-\kappa)/3} \text{ if } 0 < \kappa \leq 1.}$$

$$u_M(x) = u_0(x) - 4\pi \sum_{m=1}^{M} G(x, x_m)h(x_m)u_e(x_m)a^{2-\kappa},$$

$$|x - x_m| \geq d, \quad \kappa < 1.$$

$$d = O(a^{(2-\kappa)/3}).$$

$$
\begin{aligned}
& 4\pi \sum_{m=1}^{M} G(x, x_m) h(x_m) u_e(x_m) a^{2-\kappa} \\
= \quad & 4\pi \sum_{p=1}^{P} G(x, y^{(p)}) h(y^{(p)}) u_e(y^{(p)}) a^{2-\kappa} \sum_{x_m \in \Delta_p} 1 \\
= \quad & 4\pi \sum_{p=1}^{P} G(x, y^{(p)}) h(y^{(p)}) u_e(y^{(p)}) \frac{a^{2-\kappa}}{a^{2-\kappa}} N(y^{(p)}) |\Delta_p| (1 + \varepsilon_p) \\
\to \quad & \int_D G(x, y) p(y) u(y) dy, \quad p(y) := 4\pi h(y) N(y).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}(\Delta_p) &= \frac{1}{a^{2-\kappa}} \int_{\Delta_p} N(x) dx [1 + o(1)], \quad a \to 0. \\
u(x) &= u_0(x) - \int_D G(x, y) p(y) u(y) dy.
\end{aligned}
$$

Lemma

Lemma. If $f \in C(D)$ and x_m are distributed in D so that

$$\mathcal{N}(\triangle) = \frac{1}{\varphi(a)} \int_{\triangle} N(x)dx\,[1 + o(1)], \quad a \to 0,$$

for any subdomain $\triangle \subset D$, where $\varphi(a) \geq 0$ is a continuous, monotone, strictly growing function, $\varphi(0) = 0$, then

$$\lim_{a \to 0} \sum_{x_m \in D} f(x_m)\varphi(a) = \int_D f(x)N(x)dx.$$

Remark: *The* **Lemma** *holds for f with the set of discontinuities of Lebesgue's measure zero.*

Proof of Lemma

Proof. Let $D = \cup_p \triangle_p$ be a partition of D into a union of small cubes $\triangle_p$, having no common interior points. Let $|\triangle_p|$ denote the volume of $\triangle_p$, $\delta := \max_p diam \triangle_p$, and $y^{(p)}$ be the center of the cube $\triangle_p$. One has

$$\lim_{a\to 0} \sum_{x_m \in D} f(x_m)\varphi(a) = \lim_{a\to 0} \sum_{y^{(p)} \in \triangle_p} f(y^{(p)})\varphi(a) \sum_{x_m \in \triangle_p} 1$$

$$= \lim_{a\to 0} \sum f(y^{(p)}) N(y^{(p)}) |\triangle_p| [1 + o(1)] = \int_D f(x)N(x)dx.$$

The last equality holds since the preceding sum is a Riemannian sum for the continuous function $f(x)N(x)$ in the bounded domain D. Thus, Lemma is proved.

New equation

$$u(x) = u_0(x) - \int_D G(x,y)p(y)u(y)dy, \quad p(x) = 4\pi h(x)N(x).$$

$$L_0 u := [\nabla^2 + k^2 - q_0(x)]u = p(x)u(x), \quad Lu = 0.$$

$$L = L_0 - p(x) := \nabla^2 + k^2 - q(x), \quad q(x) = q_0(x) + p(x).$$

$$n^2(x) = 1 - k^{-2}q(x.$$

$$k^2[n_0^2(x) - n^2(x)] = p(x).$$

Creating new materials

Step 1.

$$\{n^2(x), n_0^2(x)\} \Rightarrow p(x) = k^2(n_0^2 - n^2).$$

Step 2.
Given $p(x) = p_1 + ip_2$, find $\{h(x), N(x)\}$.

Here $h(x) = h_1(x) + ih_2(x)$, $N(x) \geq 0$, $h_2(x) \leq 0$.
We have $p(x) = 4\pi N(x)h(x)$. Thus,

$$h_1(x) = \frac{p_1(x)}{4\pi N(x)}, \quad h_2(x) = \frac{p_2(x)}{4\pi N(x)}.$$

There are many solutions, because $N(x) \geq 0$ can be arbitrary.

Step 3.
Embed

$$\mathcal{N}(\Delta_p) = \frac{1}{a^{2-\kappa}} \int_{\Delta_p} N(x)dx$$

small particles in Δ_p, where $\bigcup_p \Delta_p = D$. We assume that
$\zeta_m = \frac{h(y^{(p)})}{a^\kappa}$ for all $x_m \in \Delta_p$.
The distance between neighboring particles is $d = O(a^{\frac{2-\kappa}{3}})$.
Theorem. The resulting new material has the function $n^2(x)$ as its refraction coefficient with the error which tends to 0 as $a \to 0$.
Remark. The total volume V_p of the embedded particles in the limit $a \to 0$ is zero, provided that $\kappa > -1$:

$$V_p = \lim_{a \to 0} O(a^3/a^{2-\kappa}) = \lim_{a \to 0} O(a^{1+\kappa}) = 0.$$

Spatial dispersion. Negative refraction

$$u = \sum_k a(k) e^{i[k\cdot r - \omega(k)t]}, \quad |k - \overline{k}| + |\omega(k) - \omega(\overline{k})| < \delta$$

$$v_g = \nabla_k \omega(|k|), \qquad v_p = \frac{\omega}{|k|} k^0.$$

$$\nabla_k |k| = k^0 := \frac{k}{|k|}; \qquad \frac{\omega^2 n^2}{c^2} = k^2, \qquad \frac{\omega n}{c} = |k|.$$

$$[\frac{n}{c} + \frac{\omega}{c}\frac{\partial n}{\partial \omega}]\nabla_k \omega = k^0.$$

$\{v_g = -const \cdot v_p, \quad const > 0\} \iff$ negative refraction.

$$n + \omega \frac{\partial n}{\partial \omega} < 0.$$

If $\omega > 0$, $\omega = \omega(|k|)$, then

$$v_p \cdot v_g = \omega'(|k|) \frac{\omega}{|k|} < 0$$

provided that

$$\omega'(|k|) < 0.$$

$$\nabla_k \omega(|k|) = \omega'(|k|) k^0.$$

$$\nabla_k \omega(|k|) \cdot v_p = \omega'(|k|) \frac{\omega}{|k|}.$$

Inverse scattering with data at fixed energy and fixed incident direction

$$\left[\nabla^2 + k^2 - q(x)\right] u = 0 \text{ in } \mathbb{R}^3, \quad u = e^{ik\alpha\cdot x} + v := u_0 + v,$$
$$v = A(\beta)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r = |x| \to \infty, \quad \frac{x}{r} := \beta, \tag{1}$$
$$A(\beta) = -\frac{1}{4\pi}\int_D e^{-ik\beta\cdot x} h(x)dx, \quad h(x) := q(x)u(x,\alpha).$$

IP: *Given* $f(\beta) \in L^2(S^2)$, $\alpha \in S^2$, $k > 0$, *and* $\epsilon > 0$, $D \subset \mathbb{R}^3$ (a bounded domain), *find* $q \in L^2(D)$ *such that*

$$\|f(\beta) - A(\beta)\|_{L^2(S^2)} < \epsilon. \tag{2}$$

This problem has infinitely many solutions.

Claim 1. The set $\{\int_D e^{-ik\beta\cdot x}h(x)dx\}_{\forall h\in L^2(D)}$ is dense in $L^2(S^2)$
Corollary 1. Given $f\in L^2(S^2)$ and $\epsilon>0$, one can find $h\in L^2(D)$ such that

$$\|f(\beta)+\frac{1}{4\pi}\int_D e^{-ik\beta\cdot x}h(x)dx\|<\epsilon.$$

Claim 2. The set $\{q(x)u(x,\alpha)\}_{\forall q\in L^2(D)}$ is dense in $L^2(D)$.
Corollary 2. Given $h\in L^2(D)$ and $\epsilon>0$, one can find $q\in L^2(D)$ such that

$$\|h(x)-q(x)u(x,\alpha)\|_{L^2(D)}<\epsilon.$$

Since the scattering amplitude

$$A(\beta)=-\frac{1}{4\pi}\int_D e^{-ik\beta\cdot x}h(x)dx$$

depends continuously on h, IP is solved by Claims 1,2.

Proof of Claim 1

Assume the contrary. Then $\exists\ \psi \in L^2(S^2)$ such that

$$0 = \int_{S^2} d\beta \psi(\beta) \int_D e^{-ik\beta\cdot x} h(x) dx \quad \forall h \in L^2(D).$$

Thus,

$$\int_{S^2} d\beta \psi(\beta) e^{-ik\beta\cdot x} = 0 \quad \forall x \in \mathbb{R}^3.$$

Therefore,

$$\int_0^\infty d\lambda \lambda^2 \int_{S^2} d\beta e^{-i\lambda\beta\cdot x} \psi(\beta) \frac{\delta(\lambda - k)}{k^2} = 0 \quad \forall x \in \mathbb{R}^3.$$

By the injectivity of the Fourier transform, one gets

$$\psi(\beta) \frac{\delta(\lambda - k)}{k^2} = 0.$$

Therefore, $\psi(\beta) = 0$. Claim 1 is proved.

Proof of Claim 2

Given $h \in L^2(D)$, define

$$u := u_0 - \int_D g(x,y)h(y)dy, \quad g := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{3}$$

$$q(x) := \frac{h(x)}{u(x)}. \tag{4}$$

If $q \in L^2(D)$, then this q solves the problem, and u, defined in (3), is the scattering solution:

$$u = u_0 - \int_D g(x,y)q(y)u(y)dy, \tag{5}$$

and

$$A(\beta) = -\frac{1}{4\pi}\int_D e^{-ik\beta\cdot y}h(y)dy.$$

If q is not in $L^2(D)$, then the null set $N := \{x: \ x \in D, \ u(x) = 0\}$ is non-void. Let

$$N_\delta := \{x: \ |u(x)| < \delta, x \in D\}, \quad D_\delta := D \setminus N_\delta.$$

Claim 3. $\exists h_\delta = \begin{cases} h, & \text{in } D_\delta, \\ 0, & \text{in } N_\delta, \end{cases}$ such that $\|h_\delta - h\|_{L^2(D)} < c\epsilon$,

$q_\delta := \begin{cases} \frac{h_\delta}{u_\delta}, & \text{in } D_\delta, \\ 0, & \text{in } N_\delta, \end{cases} \quad q_\delta \in L^\infty(D), \quad u_\delta := u_0 - \int_D g h_\delta dy.$

Proof. The set N is, generically, a line $l = \{x: u_1(x) = 0, \ u_2(x) = 0\}$, where $u_1 = \Re u$ and $u_2 = \Im u$. Consider a tubular neighborhood of this line, $\rho(x, l) \leq \delta$. Let the origin O be chosen on l, s_3 be the Cartesian coordinate along the tangent to l, and $s_1 = u_1$, $s_2 = u_2$ are coordinates in the plane orthogonal to l, s_j-axis is directed along $\nabla u_j|_l, \ j = 1, 2$.

The Jacobian $\mathcal{J}$ of the transformation $(x_1, x_2, x_3) \mapsto (s_1, s_2, s_3)$ is nonsingular, $|\mathcal{J}| + |\mathcal{J}^{-1}| \leq c$, because ∇u_1 and ∇u_2 are linearly independent. Define

$$h_\delta := \begin{cases} h, & \text{in } D_\delta, \\ 0, & \text{in } N_\delta, \end{cases} \qquad u_\delta := u_0 - \int_D g(x,y) h_\delta(y) dy,$$

$$q_\delta := \begin{cases} \frac{h_\delta}{u_\delta}, & \text{in } D_\delta, \\ 0, & \text{in } N_\delta. \end{cases}$$

One has $u_\delta = u_0 - \int_D g h dy + \int_D g(x,y)(h - h_\delta) dy$,

$$|u_\delta(x)| \geq |u(x)| - c \int_{N_\delta} \frac{dy}{4\pi |x-y|} \geq \delta - I(\delta), \quad x \in D_\delta, \;\; c = \max_{x \in N_\delta} |h(x)|.$$

If one proves, that $I(\delta) = o(\delta)$, $\delta \to 0$, $\forall x \in D_\delta$ then $q_\delta \in L^\infty(D)$, and Claim 3 is proved.

Claim 4:

$$I(\delta) = \mathcal{O}(\delta^2 |\ln(\delta)|), \quad \delta \to 0.$$

Proof.

$$\int_{N_\delta} \frac{dy}{|x-y|} \le \int_{N_\delta} \frac{dy}{|y|} = c_1 \int_0^{c_2\delta} \rho \int_0^1 \frac{ds_3}{\sqrt{\rho^2 + s_3^2}} d\rho$$
$$= c_1 \int_0^{c_2\delta} \rho \ln(s_3 + \sqrt{\rho^2 + s_3^2})|_0^1 \le c_3 \int_0^{c_2\delta} \rho \ln\left(\frac{1}{\rho}\right) d\rho$$
$$\le \mathcal{O}(\delta^2 |\ln(\delta)|).$$

The condition $|\nabla u_j|_l \geq c > 0, j = 1, 2$, implies that a tubular neighborhood of the line l, $N_\delta = \{x : \sqrt{|u_1|^2 + |u_2|^2} \leq \delta\}$, is included in a region $\{x : |x| \leq c_2\delta\}$ and includes a region $\{x : |x| \leq c_2'\delta\}$. This follows from the estimates

$$c_2'\rho \leq |u(x)| = |\nabla u(\xi) \cdot (x - \xi)| \leq c_2\rho.$$

Here $\xi \in l$, x is a point on a plane passing through ξ and orthogonal to l, $\rho = |x - \xi|$, and $\delta > 0$ is sufficiently small, so that the terms of order ρ^2 are negligible,
$c_2 = \max_{\xi \in l} |\nabla u(\xi)|, \quad c_2' = \min_{\xi \in l} |\nabla u(\xi)|$.

Calculation of h given $f(\beta)$ and $\epsilon > 0$

1. Let $\{\phi_j\}$ be a basis in $L^2(D)$,

$$h_n = \sum_{j=1}^{n} c_j^{(n)} \phi_j,$$

$$\psi_j(\beta) := -\frac{1}{4\pi} \int_D e^{-ik\beta\cdot x} \phi_j(x) dx.$$

Consider the problem:

$$\|f(\beta) - \sum_{j=1}^{n} c_j^{(n)} \psi_j(\beta)\| = \min. \qquad (6)$$

A necessary condition for (6) is a linear system for $c_j^{(n)}$.

2. Analytical solution:

$$D = \{x : \ |x| \leq 1\} := B, \text{ or } B \subset D, \ h = 0 \text{ in } D \setminus B.$$

One has:

$$h_{lm} = \begin{cases} (-1)^{l+1} \frac{f_{l,m}}{\sqrt{\frac{\pi}{2k}} g_{1,l+\frac{1}{2}}(k)}, & l \leq L, \\ 0, & l > L, \end{cases} \tag{7}$$

where $g_{\mu,\nu}(k) = \int_0^1 x^{\mu+\frac{1}{2}} J_\nu(kx)dx$ (Bateman-Erdelyi book, formula (8.5.8))
and L is chosen so that

$$\sum_{l>L} |f_{l,m}|^2 < \epsilon^2.$$

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Electromagnetic Wave scattering by many small bodies

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Problem formulation

$$\nabla \times E = i\omega\mu H, \quad \nabla \times H = -i\omega\epsilon'(x)E \quad \text{in } \mathbb{R}^3,$$

where $\omega > 0$, $\mu =$ const, $\epsilon'(x) = \epsilon > 0$ in $D' = \mathbb{R}^3 \setminus D$, $\quad \epsilon'(x) = \epsilon(x) + i\frac{\sigma(x)}{\omega}$; $\sigma(x) \geq 0$, $\epsilon'(x) = \epsilon > 0$ in $D' = \mathbb{R}^3 \setminus D$.

$$\nabla \times \nabla \times E = K^2(x)E, \quad H = \frac{\nabla \times E}{i\omega\mu}, \; K^2(x) := \omega^2\epsilon'(x)\mu.$$

$$E(x) = E_0(x) + v, \; E_0(x) = \mathcal{E}e^{ik\alpha\cdot x}, \quad k = \frac{\omega}{c} = \frac{1}{\sqrt{\epsilon\mu}},$$

$$v_r - ikv = o\left(\frac{1}{r}\right), \; \alpha \cdot \mathcal{E} = 0.$$

Basic equations

$$-\triangle E + \nabla(\nabla \cdot E) - k^2 E - p(x)E = 0,\ p(x) := K^2(x) - k^2 \quad (8)$$

$$\nabla \cdot (K^2(x)E) = 0 \quad (9)$$

$$-\triangle E - k^2 E - p(x)E - \nabla(q(x) \cdot E) = 0,\ q(x) := \frac{\nabla K^2(x)}{K^2(x)}.$$

$$E = E_0 + \int_D g(x,y)\Big(p(y)E(y) + \nabla_y(q(y) \cdot E(y))\Big)dy,$$

$$g(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}.$$

Assume $q = 0$ on $S = \partial D$. Then

$$E = E_0 + \int_D g(x,y)p(y)E(y)dy + \nabla_x \int_D g(x,y)q(y) \cdot E(y)dy. \quad (10)$$

Equivalence result

Lemma Equation (10) is uniquely solvable in $H^1(D)$. Its solution satisfies (8) and (9)

Proof. If E solves (10), then $(-\triangle - k^2)E = p(x)E + \nabla(q(x) \cdot E)$. Thus,

$$\nabla \times \nabla \times E - \nabla(\nabla \cdot E) - \nabla(q(x) \cdot E) = K^2(x)E,$$

or $\nabla \times \nabla \times E - \nabla \left(\frac{K^2(x)\nabla \cdot E + \nabla K^2(x) \cdot E}{K^2(x)} \right) = K^2(x)E$. Let $\frac{1}{K^2(x)} \nabla \cdot (K^2(x)E) := \psi(x)$. Then

$$-\triangle\psi - K^2(x)\psi = 0, \ K^2 = k^2 \text{in } D', \ \text{Im}K^2 \geq 0, \ \psi_r - ik\psi = o\left(\frac{1}{r}\right).$$

This implies $\psi = 0$.

Wave scattering by one small body

Let $x_m \in D$, $ka \ll 1$, $a = 0.5$ diam D. Then

$$E(x) = E_0(x) + g(x, x_m) \int_D p(y)E(y)dy + \nabla_x g(x, x_m) \int_D q(y) \cdot E(y)dy$$

$$+ \int_D [g(x, y) - g(x, x_m)]p(y)E(y)dy + \nabla_x \int_D [g(x, y) - g(x, x_m)]q(y) \cdot E(y)dy$$

$$= E_0 + E_m(x) \approx E_0 + g(x, x_m)V_m + \nabla_x g(x, x_m)\nu_m.$$

$$V_m = \frac{V_{0m}}{1 - a_m} + \frac{A_m}{1 - a_m}\nu_m, \ \ \nu_m = \frac{(1 - a_m)\nu_{0m} + B_m \cdot V_{0m}}{(1 - a_m)(1 - b_m) - B_m \cdot A_m},$$

$$V_{0m} := \int_D p(x)E_0(x)dx, \ a_m := \int_D p(x)g(x, x_m)dx, A_m := \int_D p(x)\nabla_x g(x, x_m)dx,$$

$$\nu_{0m} := \int_D q(x) \cdot E_0(x)dx, \ B_m := \int_D q(x)g(x, x_m)dx, \ b_m := \int_D q(x) \cdot \nabla_x g(x, x_m)dx.$$

Error is $O\left(ka + \frac{a}{d}\right)$.

Many-body scattering

$$E(x) = E_0(x) + \sum_{m=1}^{M} \left[g(x, x_m) V_m + \nabla_x g(x, x_m) \nu_m \right] + O\left(ka + \frac{a}{d} \right),$$

$\inf_m |x - x_m| \geq d \gg a.$

Lemma. If $f \in C(D)$ and

$$\mathcal{N}(\triangle) = \frac{1}{\varphi(a)} \int_{\triangle} N(x) dx \left[1 + o(1) \right], \quad a \to 0,$$

for any subdomain $\triangle \subset D$, where $\varphi(a) \geq 0$ is a continuous, monotone, strictly growing function, $\varphi(0) = 0$, then

$$\lim_{a \to 0} \sum_m f(x_m) \varphi(a) = \int_D f(x) N(x) dx.$$

Let D_m be a ball, centered at x_m, of radius a. Let $p(x) = p(r)$ in $D_m,\ r = |x - x_m|,\ x \in D_m,\ p(x) = 0$ in $D \setminus \cup_{m=1}^{M} D_m$,
$p(r) = \frac{\gamma_m}{4\pi a^\kappa}(1-t)^2,\ t = \frac{r}{a},\ 0 < \kappa < 3.$
$j_m := \int_{D_m} p dy = \frac{\gamma_m}{30} a^{3-\kappa},\ \gamma_m = const$ we can choose.
$Z_m := \int_{D_m} \frac{\nabla p \cdot E}{K^2 + p(y)} dy = \frac{4\pi\kappa}{3} a^3 \ln a \nabla \cdot E|_{x=x_m}[a + o(a)], a \to 0.$
Thus, $|Z_m| \ll j_m,\ a \to 0$, and, taking $\phi(a) = a^{3-\kappa}$, one gets

$$E(x) \sim E_0(x) + \phi(a) \sum_m g(x, x_m) \frac{\gamma_m}{30} E_e(x_m),\ |x - x_m| \gg a, \quad (11)$$

$$E_e(x_m) = E_e^{(m)} = E_0(x_m) + \sum_{m' \neq m} E_{m'}(x_m).$$

The limiting case

$$E(x) = E_0(x) + \int_D g(x,y)\mathcal{P}(y)E(y)dy, \quad \mathcal{P}(y) := \frac{\gamma(y)}{30}N(y),$$

$$(\nabla^2 + k^2)E + \mathcal{P}(x)E = 0,$$

$$[\nabla^2 + \mathcal{K}^2(x)]E = 0, \qquad (12)$$

$$\mathcal{K}^2(x) := k^2 + \mathcal{P}(x) := k^2 n^2(x), \ n^2(x) := 1 + k^{-2}\mathcal{P}(x)$$
$$\mathcal{K}^2(x) = \omega^2 \tilde{\epsilon}\mu.$$

Interpretation

a) $\nabla \times \nabla E = \mathcal{K}^2 E + i\omega\mu\mathcal{J}$,
$J := \sigma E = \frac{\nabla\nabla\cdot E}{i\omega\mu}, \ \sigma = \sigma_{ij} = \frac{1}{i\omega\mu}\frac{\partial^2}{\partial x_i \partial x_j}$.

b) $D(x) = \tilde{\epsilon}E(x) + \int_D \chi(x,y)E(y)dy$ (non-local susceptability)
$\nabla \times E = i\omega\mu H$,
$\nabla \times H = -i\omega\tilde{\epsilon}E - i\omega \int_D \chi(x,y)E(y)dy$.
If $\chi(x,y) := \frac{1}{\omega^2\mu}\nabla_x\delta(x-y)\nabla_y$, then
$\nabla \times \nabla \times E = \mathcal{K}^2(x)E + \mathcal{J}, \ \mathcal{J} := \frac{\nabla\nabla\cdot E}{i\omega\mu}$.

Spatial dispersion and negative refraction

$$v_g = \nabla_{\mathbf{K}} \omega(K), \; v_{ph} = \frac{\omega}{|\mathbf{K}|}\mathbf{K}^o, \; |\mathbf{K}| = K. \tag{13}$$

Negative refraction:

$$v_g \cdot v_{ph} < 0 \tag{14}$$

$$\omega n(x, \omega) = c|\mathbf{K}| \tag{15}$$

$$v_g(n + \omega \frac{\partial n}{\partial \omega}) = c\mathbf{K}^o. \tag{16}$$

If $n + \omega \frac{\partial n}{\partial \omega} < 0$, then (14) holds. $n^2 = 1 + k^{-2}\mathcal{P}(x, \omega)$.