Gravitational Radiation in the Introductory Physics Curriculum

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General Relativity (GR) is usually introduced to undergraduates in a dedicated upper level course that follows the traditional coursework in classical mechanics and electromagnetism. But even first-year students can comprehend – albeit incompletely – the astonishing discoveries in astronomy and astrophysics that are profoundly enlarging our understanding of the universe. The recent direct detection of gravitational radiation by the LIGO Scientific Collaboration provides a perfect opportunity to introduce students to the excitement of contemporary physics. With a little help from GR, students can use their understanding of Kepler's laws to reproduce the central results of the groundbreaking LIGO publication (Ref. 1).

Consider two bodies m_1 and m_2 in circular orbit about their common center of mass, separated by $a = r_1 + r_2$. General Relativity tells us that the system emits gravitational radiation at a rate

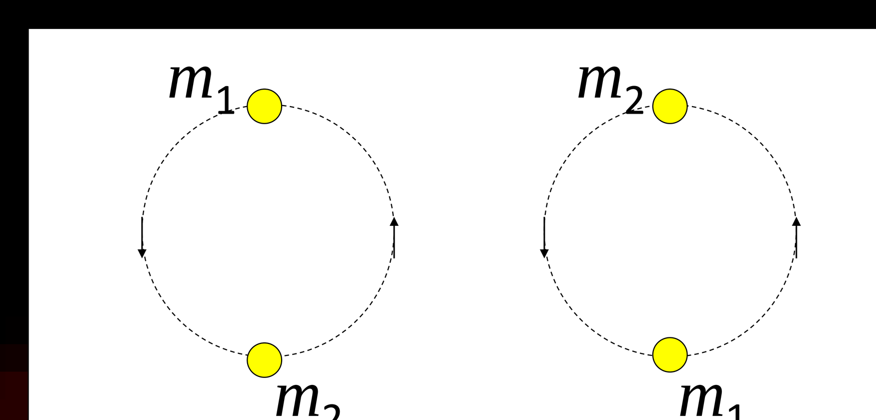
$$L = \frac{32}{5}\frac{G}{c^5}\mu^2 a^4 \omega_{orb}^6$$

where $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass, and $\omega_{orb} = 2\pi/T_{orb} = 2\pi f_{orb}$ is the orbital angular velocity of either body. The total mechanical energy of the system is (as usual) half the potential energy: $E = -Gm_1m_2/2a$. Setting $dE/dt = -L$, we obtain

$$\frac{1}{T_{orb}}\frac{dT_{orb}}{dt} = -\frac{96}{5}\frac{G^3}{a^4c^5}\frac{\mu^2(m_1+m_2)^3}{m_1m_2} \quad .$$

The LIGO signal allows us to measure T_{orb} and dT_{orb}/dt, allowing us to draw conclusions about the mass and nature of the system.

The period (or frequency) of the emitted gravitational wave is directly related to the orbital period (or frequency) of the binary system: $T_{gw} = \frac{1}{2}T_{orb}$ so $f_{gw} = 2f_{orb}$. This relation is easy to understand in the case of two identical bodies. When they execute *half* an orbit, they exchange places, and the gravitational effects – including radiation – are the same as at the beginning of the orbit. (See the figure to the right.)



We cannot determine the orbital radius a from the LIGO signal, but we can use Kepler's 3rd law to eliminate a from the above equation:

$$a^4 = \left(\frac{T_{orb}^2}{4\pi^2}G(m_1+m_2)\right)^{4/3}$$

and after much messy algebra (which might be left as a student exercise), we find

$$\frac{1}{T_{orb}}\frac{dT_{orb}}{dt} = -\frac{96}{5c^5}G^{5/3}\mathcal{M}^{5/3}(2\pi)^{8/3}T_{orb}^{-8/3}$$

where $\mathcal{M} = \frac{(m_1m_2)^{3/5}}{(m_1+m_2)^{1/5}}$ is called the *chirp mass*. Finally, using $\frac{1}{T_{orb}}\frac{dT_{orb}}{dt} = -\frac{1}{f_{orb}}\frac{df_{orb}}{dt}$ and $f_{gw} = 2/T_{orb}$ we obtain

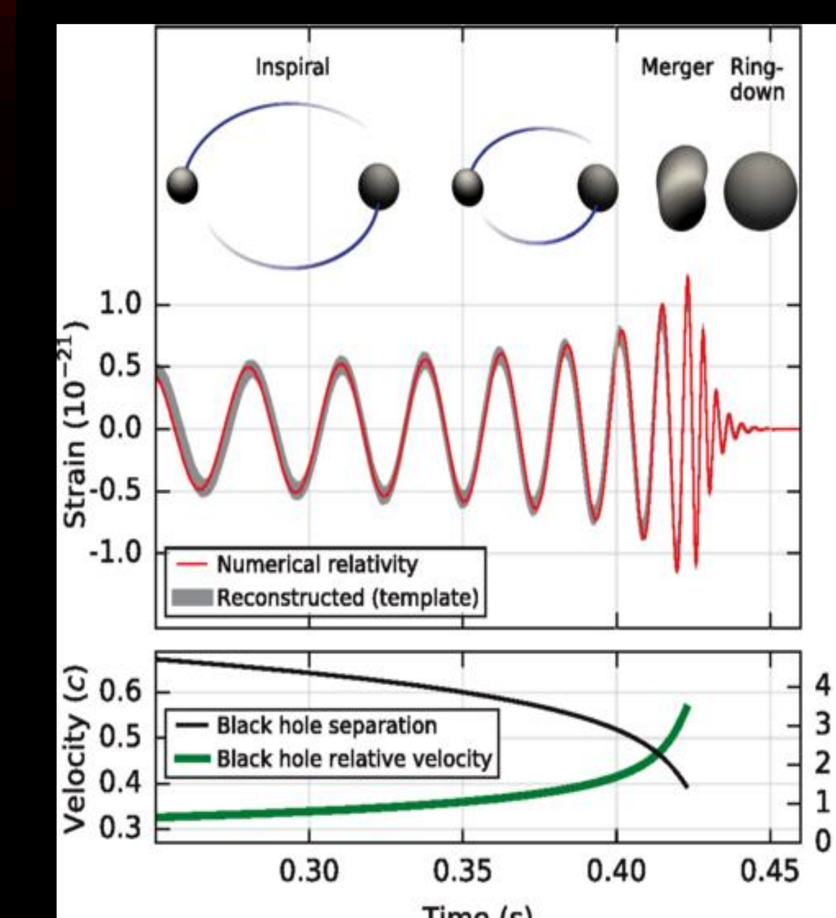
$$\frac{1}{f_{gw}}\frac{df_{gw}}{dt} = \frac{96}{5c^5}G^{5/3}\mathcal{M}^{5/3}\pi^{8/3}f_{gw}^{8/3}$$

Solving for the chirp mass,

$$\mathcal{M} = \frac{c^3}{G}\left(\frac{5}{96}\pi^{-8/3}f^{-11/3}\frac{df}{dt}\right)^{3/5}$$

In the last equation, we have dropped the subscript "gw" to conform to the notation used in the literature. *This is the result we will work with.*

The figure below is taken from Ref 1, which states that the frequency of the gravitational wave increases from 35 to 150 Hz during the interval $\Delta t = t_2 - t_1 = .425 - .250 = .175$ s. *Here are some exercises that students should be able to complete successfully.*



1. Let $A = \frac{c^5}{G^{5/3}}\frac{5}{96}\pi^{-8/3} \simeq 5.45\times10^{56}$ in SI units.

   Writing $\mathcal{M}^{5/3} = Af^{-11/3}\,df/dt$ integrate over Δt to obtain

   $$\mathcal{M}^{5/3}\Delta t = A\int_{t_1}^{t_2} f^{-11/3}df = -\frac{3}{8}A\,f^{-8/3}\Big|_{t_1}^{t_2}$$

   Use the LIGO data to calculate $\mathcal{M}$ (Ans: 30 M_{Sun})

2. Show that the total mass must be greater than about 70 M_{Sun}. (Hint: let $m_2 = \alpha m_1$ and derive expressions for $\mathcal{M}$ and M. Then minimize M.)

3. Just before the two bodies merged, their orbital frequency was about 75 Hz. Estimate the separation between the bodies at this time, assuming $M = 70M_{Sun}$. (Ans: 350 km)

4. Assume one of the bodies was a neutron star ($m = 1.4M_{Sun}$). Given $\mathcal{M} = 30M_{Sun}$, what was the mass of the other body? (Ans: 3 x 10^3 M_{Sun})

5. Calculate the Schwarzchild radius $R_S = 2GM/c^2$ for the masses used in Q.4. Assuming the bodies merge when their separation equals R_S, what would have been the gravitational wave frequency just before they merged? (Ans: 7.6 Hz)

6. Clearly, the two bodies must have been compact bodies, either black holes or neutron stars. Why did the LIGO team conclude that they were both black holes?

References: B. P. Abbott *et al.*, *Phys. Rev. Lett.* **116**, 061102 (2016)