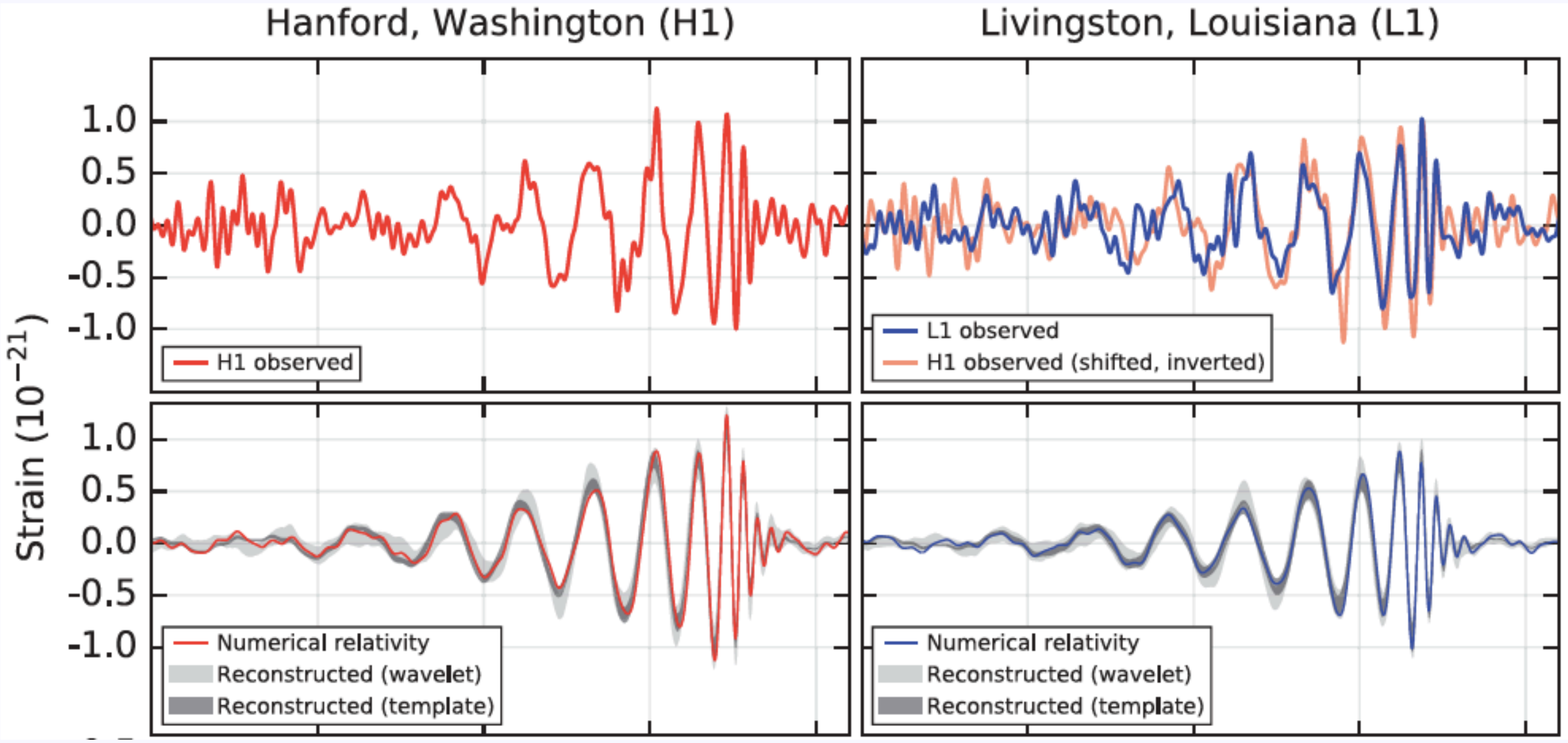
Exploring Black Holes at a Liberal Arts College

Thomas Baumgarte

Bowdoin College

Gravitational Waves!



[Abbott $et. al.$, Feb. 11, 2016]

Topics

- Numerical Relativity
  - Why does the formulation of the equations matter?
  - Why do “trumpet” coordinates help black-hole simulations?
- Research with undergraduate students
  - What is a suitable project?
  - What background do students need?
- Teaching GR at an undergraduate institution
  - How does course affect research opportunities?
  - How does research affect teaching?

Numerical Relativity

Solve Einstein's equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

on the computer!

Numerical Relativity

Solve Einstein's equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

on the computer!

Start with scalar wave equation

$$-\partial_t^2 \psi + \nabla^2 \psi = 4\pi\rho$$

Define $\kappa \equiv -\partial_t \psi$ to write as pair

$$\begin{aligned} \partial_t \psi &= -\kappa \\ \partial_t \kappa &= -\nabla^2 \psi + 4\pi\rho \end{aligned}$$

- can integrate numerically without any problems

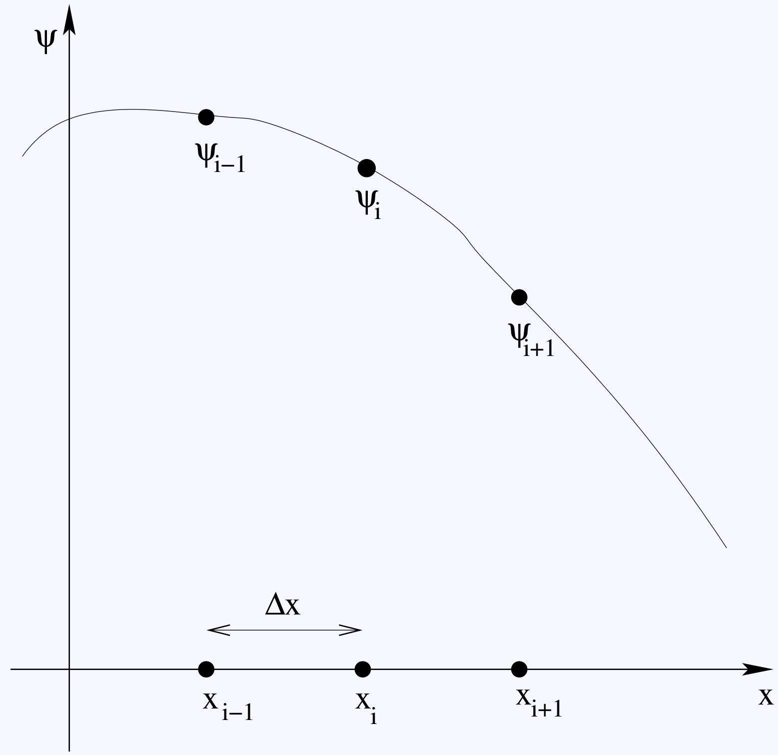
Computational Interlude: "Finite-differencing"

- Consider 1D grid with $x_i = i\Delta x$
- Represent $\psi(x,t)$ on grid: $\psi_i(t) = \psi(x_i, t)$
- "Finite-difference" first derivative

$$\left.\frac{\partial \psi}{\partial x}\right|_{i+1/2} = \frac{\psi_{i+1} - \psi_i}{\Delta x}$$

- Second derivative: first derivative of first

$$\begin{aligned}\left.\frac{\partial^2 \psi}{\partial x^2}\right|_i &= \frac{1}{\Delta x}\left(\left.\frac{\partial \psi}{\partial x}\right|_{i+1/2} - \left.\frac{\partial \psi}{\partial x}\right|_{i-1/2}\right) \\ &= \frac{\psi_{i+1} - 2\psi_i + \psi_{i-1}}{\Delta x^2}\end{aligned}$$



- insert into wave equation

$$\begin{aligned}\partial_t \psi_i &= -\kappa_i \\ \partial_t \kappa_i &= -\tfrac{1}{\Delta x^2}(\psi_{i+1} - 2\psi_i + \psi_{i+1}) + 4\pi\rho_i\end{aligned}$$

$\Longrightarrow$ Turned PDE for ψ and κ into system of coupled ODEs for ψ_i and κ_i

- Solve with Runge-Kutta ("Method of lines"), or finite-difference in time as well

Numerical Relativity

Solve Einstein's equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

on the computer!

Start with scalar wave equation

$$-\partial_t^2 \psi + \nabla^2 \psi = 4\pi\rho$$

Define $\kappa \equiv -\partial_t \psi$ to write as pair

$$\boxed{\begin{aligned} \partial_t \psi &= -\kappa \\ \partial_t \kappa &= -\nabla^2 \psi + 4\pi\rho \end{aligned}}$$

- can integrate numerically without any problems
- equations *symmetric hyperbolic*

$\Longrightarrow$ problem "well-posed"

$\Longrightarrow$ can prove that solutions are "well-behaved": don't grow faster than exponentially

Electromagnetism (E&M)

Maxwell's equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \partial_t \mathbf{E} = \nabla \times \mathbf{B} - 4\pi \mathbf{j}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \partial_t \mathbf{B} = -\nabla \times \mathbf{E}$$

Rewrite in terms of *vector potential* $\mathbf{A}$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Then

- constraint $\nabla \cdot \mathbf{B} = 0$ satisfied automatically
- curl of $\mathbf{B}$ becomes $\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$
- Also: $\partial_t \nabla \times \mathbf{A} = -\nabla \times \mathbf{E}$ implies

$$\partial_t \mathbf{A} = -\mathbf{E} + (\text{some vector whose curl vanishes}) = -\mathbf{E} - \nabla\phi$$

$\Longrightarrow$ *evolution equations:*

$$\boxed{\begin{aligned} \partial_t \mathbf{A} &= -\mathbf{E} - \nabla\phi \\ \partial_t \mathbf{E} &= -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) - 4\pi \mathbf{j} \end{aligned}}$$

Let's compare...

$$\partial_t \psi = -\kappa$$
$$\partial_t \kappa = -\nabla^2 \psi + 4\pi\rho$$

$$\partial_t \mathbf{A} = -\mathbf{E} - \nabla\phi$$
$$\partial_t \mathbf{E} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) - 4\pi\mathbf{j}$$

- new gauge variable ϕ: can choose arbitrarily – not determined by Maxwell's eqs.
- new second term involving second spatial derivatives:

$$-\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = -\nabla^k \nabla_k A_i + \nabla_i \nabla_k A^k$$

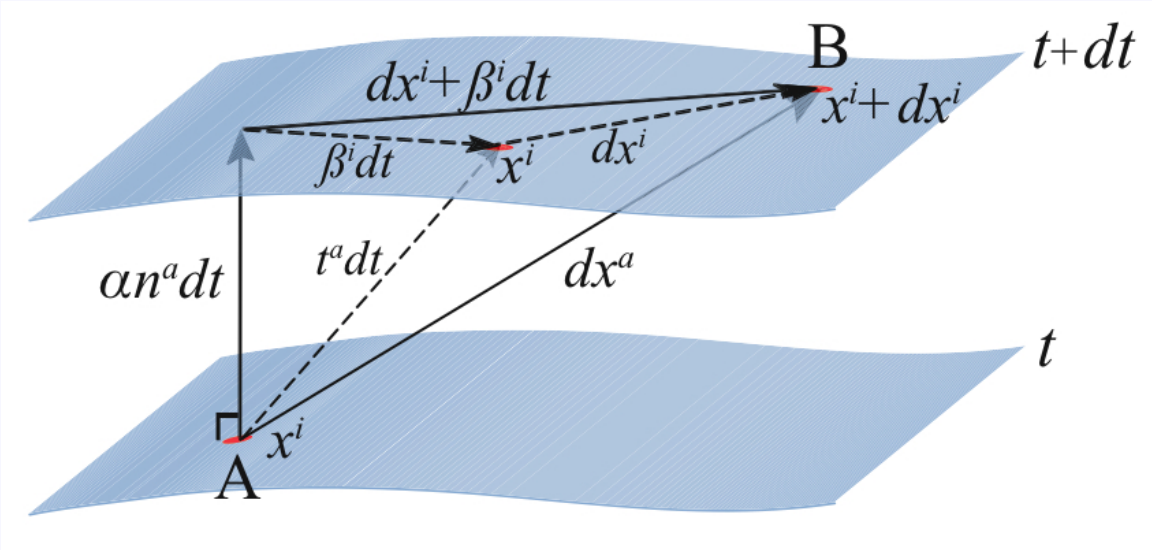
  Laplace operator plus "offending term"
  $\Longrightarrow$ in general, this spoils symmetric hyperbolicity (not wave equation for A_i)
  $\Longrightarrow$ cannot prove well-posedness
- could choose *Coulomb gauge* $\nabla \cdot \mathbf{A} = 0$ ($\Longleftrightarrow \nabla^2\phi = -4\pi\rho$) to restore hyperbolicity, but this uses up gauge freedom
- Also have one constraint in E&M,

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

  (has to be satisfied by initial data)

General Relativity

- spacetime metric $g_{\alpha\beta}$ measures proper distances in spacetime
- can "foliate" 4D spacetime into stack of spatial slices: "3+1 decomposition"



$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- spatial metric γ_{ij} measures proper distance within spatial slice of constant t
- Lapse α and shift vector β^i encode coordinate freedom
- Can choose α and β^i freely – not determined by Einstein's equations

The 3+1 decomposition of Einstein's equations

Einstein's equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

form 10 equations for 10 independent components of spacetime metric $g_{\mu\nu}$

- but can freely choose 4 of them (coordinate freedom)

$\Longrightarrow$ 4 of the 10 equations must be redundant

Einstein's equations split into

Constraint equations
- constrain γ_{ij} within each slice
- solve to construct *initial data*
- like "div" equations in E&M

Evolution equations
- evolve γ_{ij} from one slice to next
- solve to study *evolution*
- like "curl" equations in E&M

$\Longrightarrow$ "ADM" formulation [Arnowitt, Deser & Misner, 1962; York 1979]

The ADM equations

Evolution equations have very similar structure again:

$$\begin{aligned}\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_{(i}\beta_{j)} \\ \partial_t K_{ij} &= \alpha R_{ij} - 8\pi\alpha M_{ij} + \text{non-linear terms} + \text{gauge terms}\end{aligned}$$

where

- K_{ij} is *extrinsic curvature*; measures time derivative of spatial metric
- R_{ij} is *Ricci tensor*; measures curvature, contains spatial derivatives of γ_{ij}

$$R_{ij} = -\frac{1}{2}\gamma^{kl}(\partial_k\partial_l\gamma_{ij} + \partial_i\partial_j\gamma_{kl} - \partial_i\partial_l\gamma_{kj} - \partial_k\partial_j\gamma_{il}) + \ldots$$

  Laplace-operator term plus “offending” mixed second derivatives
- M_{ij} are matter terms

Let's compare again...

Scalar Wave:

$$\begin{aligned}\partial_t \psi &= -\kappa \\ \partial_t \kappa &= -\nabla^2 \psi + 4\pi\rho\end{aligned}$$

E&M:

$$\begin{aligned}\partial_t A_i &= -E_i - \nabla_i \phi \\ \partial_t E_i &= -\nabla^k \nabla_k A_i + \nabla_i \nabla^k A_k - 4\pi j_i\end{aligned}$$

GR:

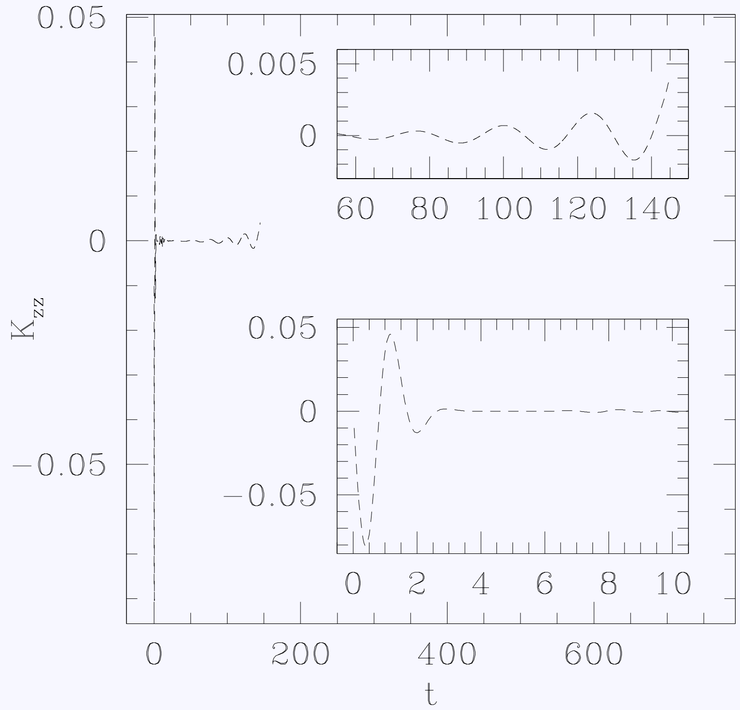
$$\begin{aligned}\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_{(i}\beta_{j)} \\ \partial_t K_{ij} &= \alpha R_{ij} - 8\pi\alpha M_{ij} + \text{non-linear terms} + \text{gauge terms}\end{aligned}$$

	Scalar Wave	E&M	GR
variables	rank 0	rank 1	rank 2
gauge variables	no	rank 0	rank 1
constraints	no	rank 0	rank 1

Integrate ADM equations numerically...

...and terrible things will happen!

- for example, consider small-amplitude wave
- adopt simple coordinate condition
  (Here: "harmonic slicing")
- code crashes after short time
- adding black holes doesn't make it easier...



$\Longrightarrow$ Consider formulation of the evolution equations...

Reformulating Einstein's equations

Can reformulate evolution equations by

- introducing new auxiliary variables
- adding constraints to evolution equations

How can this affect numerical behavior???

- mathematical properties:
  - constraints contain as high derivatives as evolution equations
  - adding constraints to evolution equations affects principle operators

  $\Longrightarrow$ affects hyperbolicity

  $\Longrightarrow$ affects well-posedness...

- for heuristic argument, consider constraint violations:
  - exact solutions must satisfy all reformulations of evolution equations,
  - but constraint violations may behave very differently

Evolution of constraint violations in electromagnetism

Recall Maxwell's evolution equations

$$\begin{aligned}\partial_t \mathbf{A} &= -\mathbf{E} - \nabla\Phi \\ \partial_t \mathbf{E} &= -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) - 4\pi \mathbf{j}\end{aligned}$$

with constraint equation

$$\nabla \cdot \mathbf{E} = 4\pi\rho.$$

Define constraint violation

$$\mathcal{C} \equiv \nabla \cdot \mathbf{E} - 4\pi\rho$$

Exercise 1

Show that the constraint violations $\mathcal{C}$ satisfy

$$\partial_t \mathcal{C} = 0.$$

Reformulating Maxwell's equations

Introduce auxiliary variable: "divergence of green variable"

$$\Gamma \equiv \nabla \cdot \mathbf{A}$$

Then

$$\begin{aligned} \partial_t \mathbf{A} &= -\mathbf{E} - \nabla \Phi \\ \partial_t \mathbf{E} &= -\nabla^2 \mathbf{A} + \nabla \Gamma - 4\pi \mathbf{j} \end{aligned}$$

$\Longrightarrow$ "offending" second derivatives absorbed in $\nabla\Gamma$

Use constraint $\nabla \cdot \mathbf{E} = 4\pi\rho$ in new evolution equation for Γ

$$\partial_t \Gamma = \partial_t \nabla \cdot \mathbf{A} = -\nabla \cdot \mathbf{E} - \nabla^2 \Phi = -4\pi\rho - \nabla^2 \Phi$$

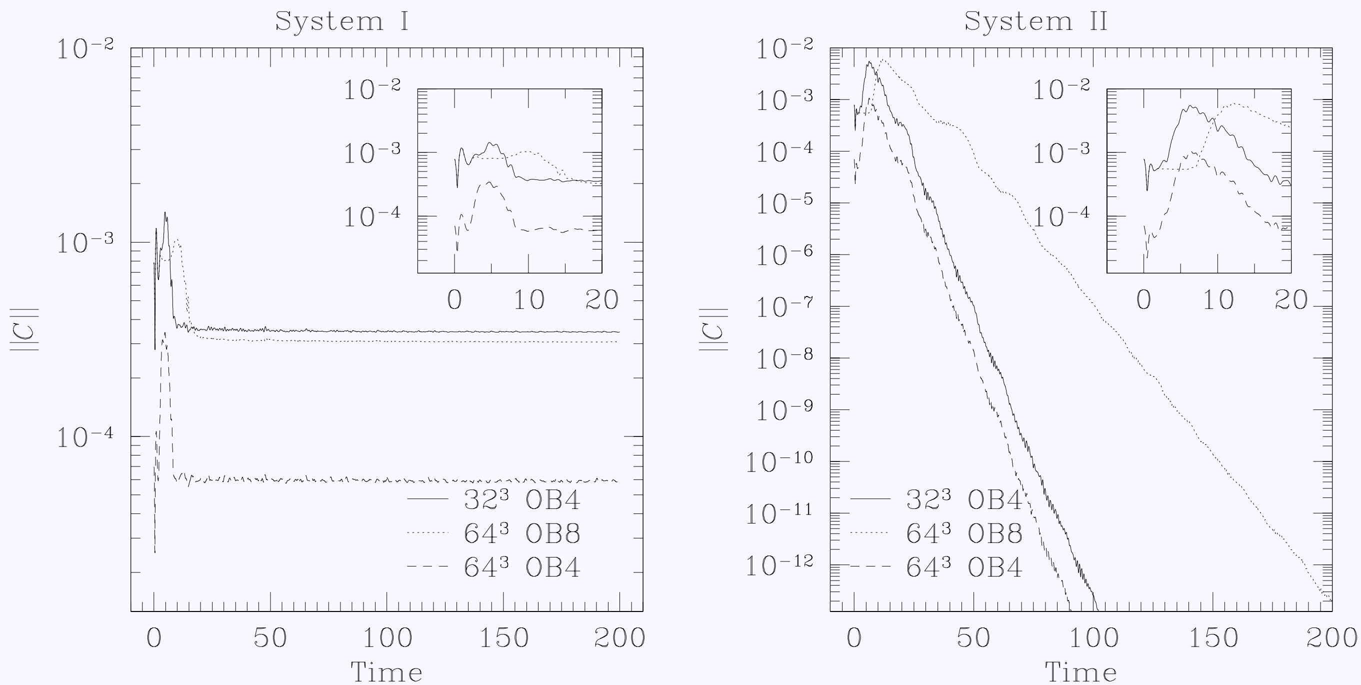
Exercise 2

Show that the constraint violations $\mathcal{C}$ now satisfy the wave equation

$$(-\partial_t^2 + \nabla^2)\,\mathcal{C} = 0.$$

Numerical Exploration

- Evolve both formulations of Maxwell's equations numerically
- Compare constraint violations



[Knapp *et.al.*, 2002]

$\Longrightarrow$ formulation of equations affects behavior of error
$\Longrightarrow$ affect stability of numerical implementations

Undergraduate Research

- Undergraduate research project with Andrew Knapp '03 and Eric Walker '03 [PRD **65**, 064031 (2002)]
- Well suited as undergraduate project because ...
  - ... independent project
  - ... limited in scope
  - ... accessible given students' background
  - ... interesting, but not too interesting...
- This project did not even require GR
  - AAPT Workshop *Teaching GR to Undergraduates* (Syracuse, July 2006)
  - started offering GR course in 2007 (includes differential geometry)
  - prepares students for research
  - can now assign GR projects

Reformulations of Einstein's equations

Can use very similar manipulations to reformulate ADM equations...

For example...

- perform conformal transformation

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$$

- introduce "conformal connection functions"

$$\bar{\Gamma}^i = -\partial_j \bar{\gamma}^{ij}$$

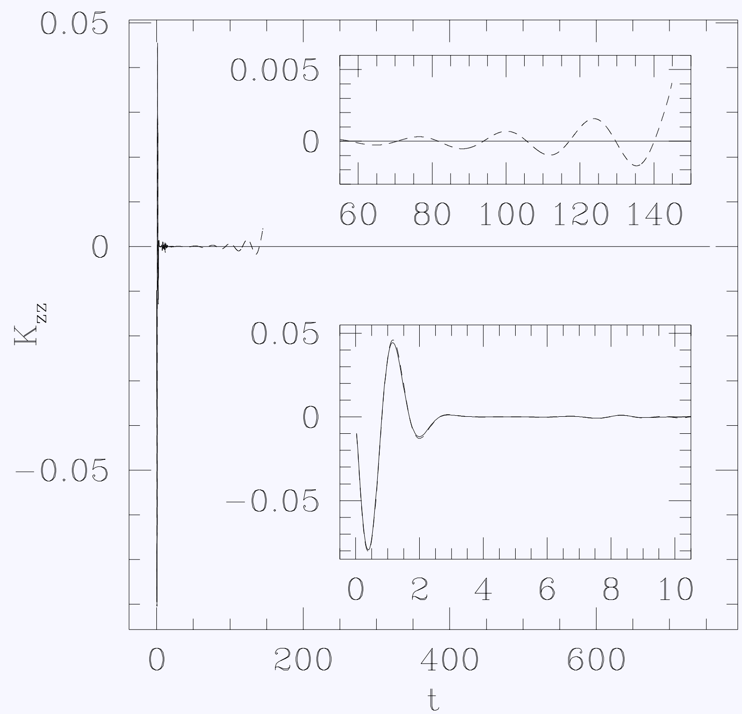
("divergence of green variable")

- these absorb "offending" second derivatives
- Use constraint in evolution equation for $\bar{\Gamma}^i$

$\Longrightarrow$ much improved numerical behavior

$\Longrightarrow$ "BSSN" formulation

[Nakamura *et.al.*, 1987; Shibata & Nakamura, 1995; Baumgarte & Shapiro; 1998]



[Baumgarte & Shapiro, 1998]

Successful reformulations of Einstein's equations

- "BSSN" formulation
  [Nakamura $et.al.$, 1987; Shibata & Nakamura, 1995; Baumgarte & Shapiro; 1998]
- Generalized harmonic formulation
  [Friedrich, 1985; Garfinkle, 2002; Pretorius, 2005; Lindblom $et.al.$, 2006]
- Z4 formulation
  [Bona $et.al.$, 2003; Bernuzzi & Hilditch, 2010]
- Fully Constrained Formulation
  [Bonazzolla $et.al.$, 2004; Cordero-Carrión $et.al.$, 2011]

All involve Γ-like quantities that absorb mixed second derivatives of metric

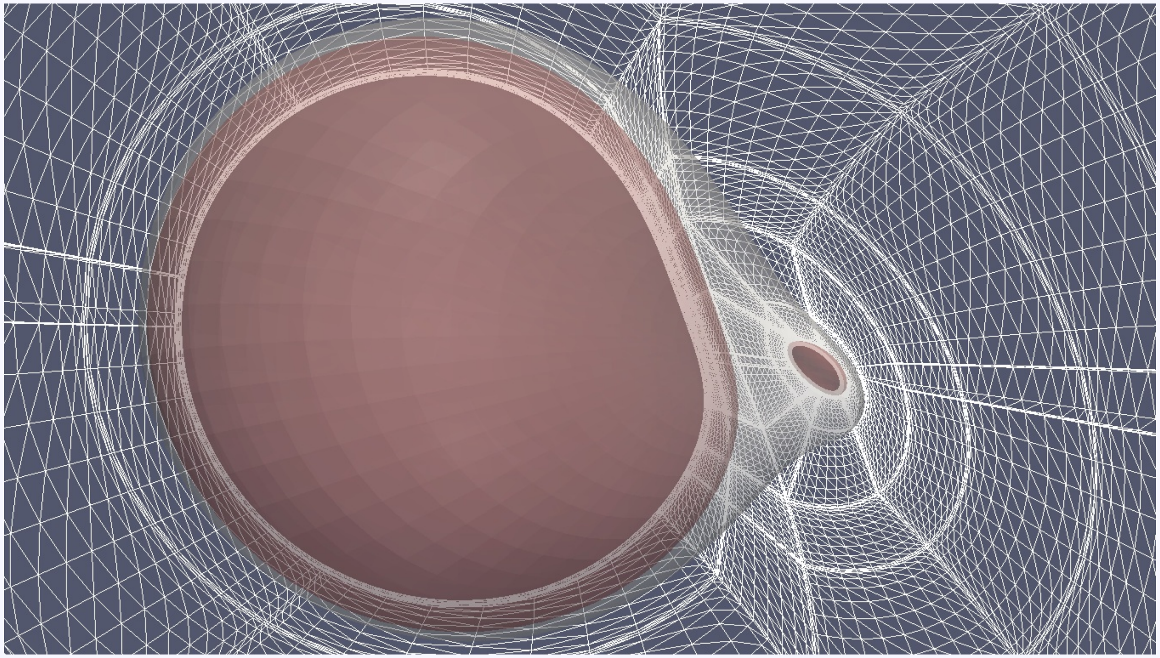
[De Donder, 1921; Lanczos, 1922]

Treating the black hole singularity

- Black hole excision:
  - ◦ no information can propagate from inside black hole to outside

  $\Longrightarrow$ excise black hole interior from numerical grid

  [Unruh, 1984; Seidel & Suen, 1992; Alcubierre & Brügmann, 2001; Pretorius, 2005]



[Caltech/Cornell group, Scheel *et.al.*]

Moving-puncture method

Can achieve similar effect with coordinate condition
[Campanelli *et.al.*, 2006; Baker *et.al.*, 2006]

Choose...

- ... 1+log slicing for lapse α

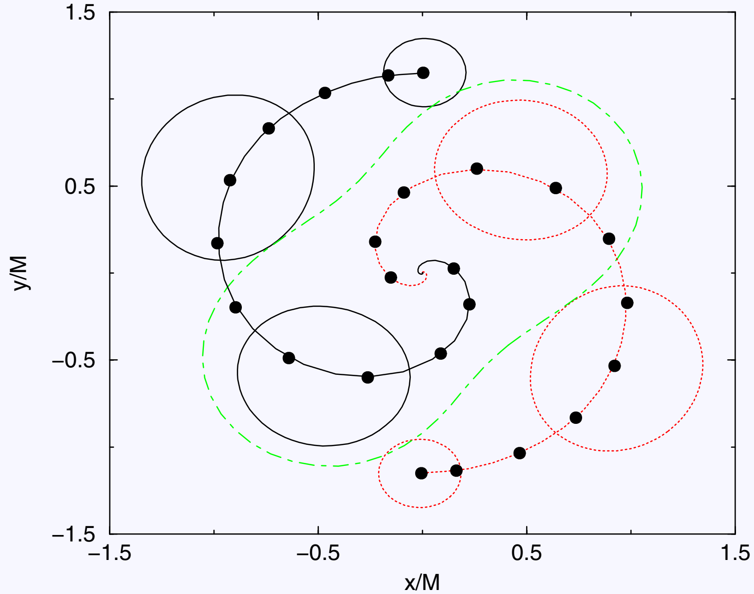
$$(\partial_t - \beta^i \partial_i)\alpha = -2\alpha K$$

[Bona *et.al.*, 1995]

- ... $\bar{\Gamma}^i$-driver condition for shift β^i

$$\bar{\Gamma}^i \sim \text{constant}$$

[Alcubierre *et.al.*, 2003]



[Campanelli *et.al.*, 2006]

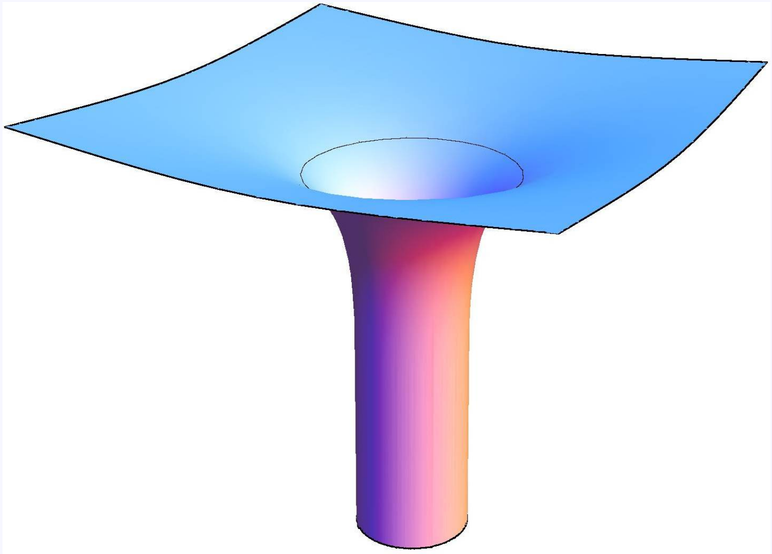
... and all works fine!?!

Trumpet geometries

Evolve Schwarzschild with moving-puncture coordinates [Hannam $et.al.$, 2006]

- induces coordinate transition
- settles down to slice with interesting properties:
  - "puncture" at $r = 0$: conformal factor ψ diverges
  - lapse vanishes at $r = 0$
  - sphere of radius $r = 0$ has non-zero proper area
  - points at $r = 0$ have infinite proper distance from $r = \epsilon$

$\Longrightarrow$ "trumpet geometry"



- Exact form depends on version of moving-puncture condition [Hannam $et.al.$, 2007]
- For some can obtain analytical expressions [Baumgarte & Naculich, 2007]

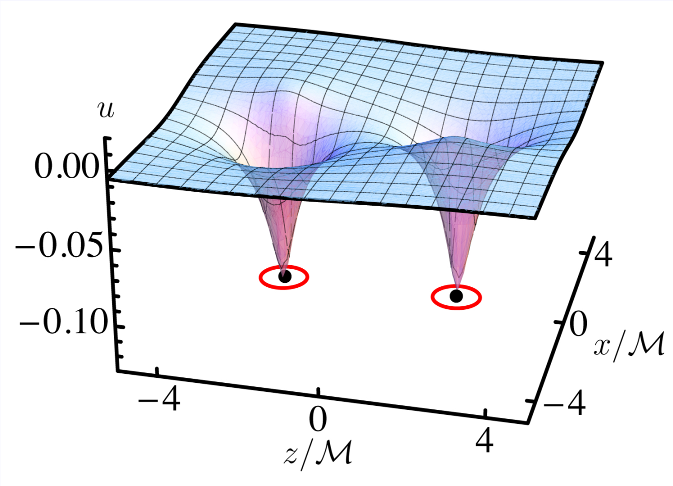
Treasure chest of undergraduate projects...

Jason Immerman '10:
Trumpet-puncture initial data for black holes
[PRD **80**, R061501 (2009)]

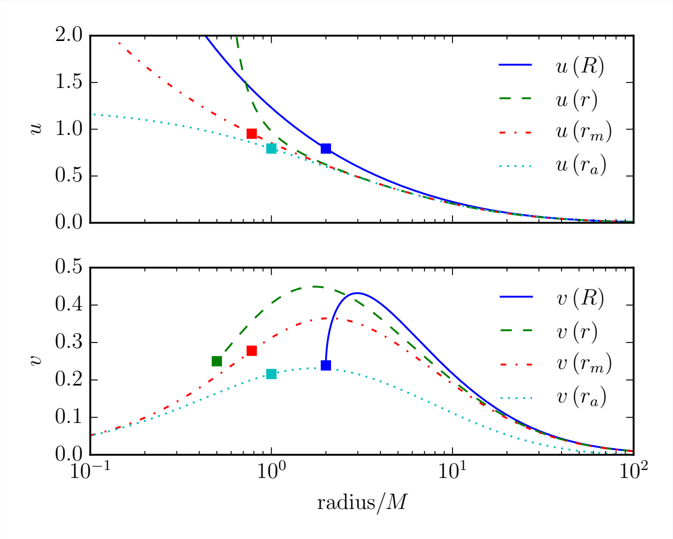
John Wendell '11:
Trumpet slices of the Schwarzschild-Tangherlini spacetime
[PRD **82**, 124057 (2010)]

Alexa Staley '11:
Oppenheimer-Snyder collapse in moving-puncture coordinates
[CQG **29**, 015003 (2012)]

August Miller '16:
Bondi accretion in trumpet geometries



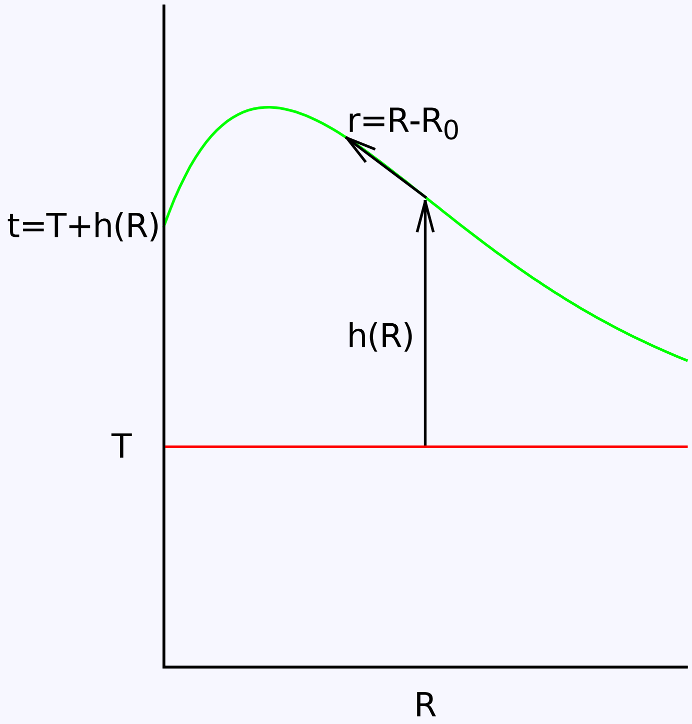
[Immerman & Baumgarte, 2009]



[Miller & Baumgarte, in prep]

Analytical trumpet geometries

Perform coordinate transformation from Schwarzschild coordinates T, R to new coordinates t, r in two steps:



- new time coordinate

$$t = T + h(R)$$

- new radial coordinate inside new slice

$$r = r(R)$$

Here: choose

$$r = R - M$$

and make spatial metric isentropic: $\gamma_{ij} = \psi^4 \eta_{ij}$

$\Longrightarrow$ find remarkably simple metric

$$\boxed{ds^2 = -\frac{r-M}{r+M}dt^2 + \frac{2M}{r}dtdr + \left(1+\frac{M}{r}\right)^2 \left(dr^2 + r^2 d\Omega^2\right)}$$

[Dennison & Baumgarte, 2014]

Let's explore...

$$ds^2 = -\frac{r-M}{r+M}\,dt^2 + \frac{2M}{r}dtdr + \left(1+\frac{M}{r}\right)^2 \left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right)$$

or

$$\alpha = \frac{r}{r+M} \qquad \beta^r = \frac{rM}{(r+M)^2} \qquad \psi = \left(1+\frac{M}{r}\right)^{1/2}$$

Horizon at $r = M$ (recall $r = R - M$)

- regular on horizon at $r = M$ ("horizon-penetrating coordinates")
- lapse α vanishes at $r = 0 \Longrightarrow$ slice covers $R \geq M$ only
- proper length of circle in equatorial plane

$$\ell = \int_0^{2\pi} (\gamma_{\varphi\varphi})^{1/2} d\varphi = \int_0^{2\pi} \psi^2 r d\varphi = 2\pi(r+M) = 2\pi R$$

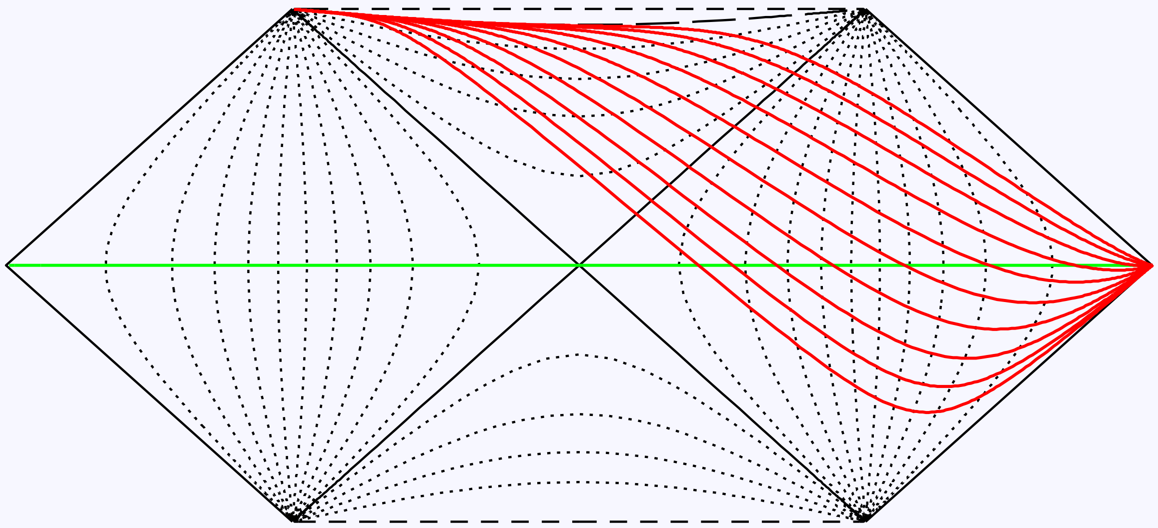
  $\Longrightarrow$ finite as $r \to 0$
- proper distance from $r = 0$ to $r = \epsilon$

$$\ell = \int_0^{\epsilon} (\gamma_{rr})^{1/2} dr = \int_0^{\epsilon} \psi^2 dr \approx M \int_0^{\epsilon} \frac{dr}{r} = M\left[\ln r\right]_0^{\epsilon} \to \infty$$

$\Longrightarrow$ new coordinates represent trumpet geometry

Properties of trumpet geometries

- don't reach spacetime curvature singularity at $R = 0$
- connect spatial infinity with future timelike infinity
- feature *coordinate* singularity at puncture $r = 0$, but infinite proper distance from all other points
- codes can handle this, as long as no grid-point at puncture



Great material for GR course

- example of simple coordinate transformation
- regular on horizon
- explore features of trumpet geometries
- important insights for numerical relativity
- can be generalized for Kerr black holes
  $\Longrightarrow$ formalism for characterizing trumpet geometries in non-spherical spacetimes [Dennison *et.al.*, 2014]

2 The Schwarzschild metric is

$$ds^2 = -f_0 dt^2 + f_0^{-1} dR^2 + R^2 d\Omega^2, \tag{2}$$

where we have abbreviated

$$f_0 = 1 - \frac{2M}{R} = \frac{R-2M}{R}. \tag{3}$$

(a) Consider a coordinate transformation to a new time coordinate $\bar{t}$ that satisfies

$$d\bar{t} = dt + \frac{1}{f_0}\frac{M}{R-M} dR. \tag{4}$$

Express the line element (2) in terms of the coordinates $\bar{t}$ and R (as well as the angular coordinates, of course). (*Hint:* You should find a non-zero "off-diagonal" term that involves the product $d\bar{t}dR$.)
(b) Now consider a new (isotropic) radial coordinate $r = R - M$, and show that you can express the line element (2) in the form

$$ds^2 = -A d\bar{t}^2 + 2B d\bar{t} dr + \psi^4 (dr^2 + r^2 d\Omega^2). \tag{5}$$

Find the coefficients A, B and ψ as functions for r only. (*Check*: If all goes well, you should find $B = M/r$.)

Summary

- Some key ingredients in numerical relativity simulations
  - formulation of equations
  - handling of black hole singularities: trumpet geometries
- Undergraduate research projects
  - provide valuable research experience for student
  - provide useful and interesting results
  - require training: GR course
- Teaching
  - GR course at Bowdoin prepares students for research projects
  - Research interests affect topics covered in class