Physics Students' Use of Layers and Representations to Understand Integrals

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1. Research Questions

- RQ1: How can we apply the mathematical "layers" framework to physics education?
- RQ2: How can we use this framework to develop instructional materials?
- RQ3: How can we use this framework to assess student work?

2. The Layers Framework

- Sample integral: $\Delta x = \int v\, dt$
- Layered version: $\Delta x = dx_1 + dx_2 + dx_3 + \cdots = v_1\, dt + v_2\, dt + v_3\, dt + \cdots$

Sum layer ($dx_1 + dx_2 + dx_3$) — Product layer ($v_1\, dt$) — Sum layer ($v_1\, dt + v_2\, dt + v_3\, dt$)

Students understand the integral in terms of layers when they view it as a sum of products. Many other views of the integral are possible: the integral as the area under a curve, or as a transformation from one equation (x^2) to another ($\frac{1}{3}x^3$). These other views do not imply an understanding in terms of layers.

3. Motivation Behind the Layers Framework

The framework has been developed by Zandieh[1], Thompson and Silverman[2], and Sealey[3]. One argument in favor of the layers framework is that it is connected with the Riemann Sum, and therefore it can be viewed as underlying the other views of integration (See box 2). For instance, it is possible to use a layered approach to show why the integral is the area under the curve (see box 5, "graphical"), or why the integral of x^2 is $\frac{1}{3}x^3$ (since the product layer $x^2\, dx$ is equal to the increment of $\frac{1}{3}(x+dx)^3 - \frac{1}{3}x^3 = x^2\, dx + \ \ldots$). If we care that students understand the connections between the various views of integration – perhaps because we want them to view scientific knowledge as being generally coherent – then the layers framework may help us achieve this.

1. M. Zandieh, A theoretical framework for analyzing student understanding of the concept of derivative, in *Research in Collegiate Mathematics Education IV*
2. P. W. Thompson and J. Silverman, The concept of accumulation in calculus, in *Making the Connection: Research and Teaching in Undergraduate Mathematics*
3. V. Sealey, in *Proceedings of the Twenty Eighth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education 2*

4. Sample Instruction (Debate Problem)

(Starts with a story about a falling coffee filter, which students have seen in lab.) A few students are talking about finding an equation for the *total* displacement of the coffee filter as it falls. Which ones, if any, are right? Which ones, if any, are wrong? Or are they partly right and partly wrong? How would you convince people when they are wrong?

a. Charles: "The equation is: $\Delta h \approx v_{total} t_{total}$.
This is a quick way of summarizing the whole series of numbers."

b. Sandy: "The equation is: $\Delta h \approx h_6 + v_1\, dt_1 + v_2\, dt_2 + v_3\, dt_3 + v_4\, dt_4 + v_5\, dt_5$.
We are adding up the little displacements from the different small amounts of time."

c. Matt: "The equation is: $\Delta h \approx h_1 - v_1\, dt_1 - v_2\, dt_2 - v_3\, dt_3 - v_4\, dt_4 - v_5\, dt_5$.
The velocities are negative because they point down."

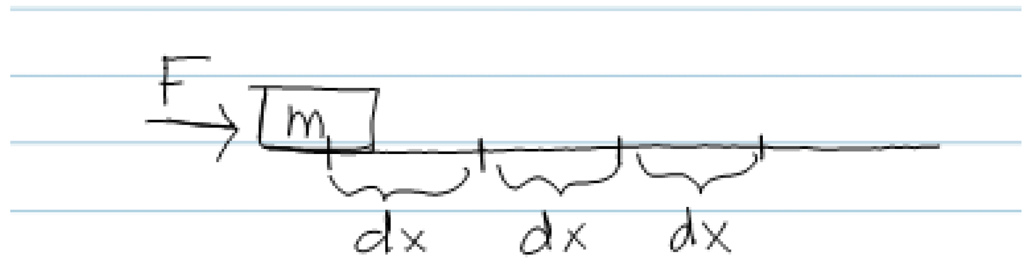
d. Jackie: "The equation is: $\Delta h \approx t_1\, dv + t_2\, dv_2 + t_3\, dv_3 + t_4\, dv_4 + t_5\, dv_5$.
This is basically the same as Sandy's equation, except it emphasizes the changing velocity."

e. Dawn: "The equation is: $\Delta h \approx \Sigma v_i\, dt_i$.
This is just a shorthand way of writing the sum."

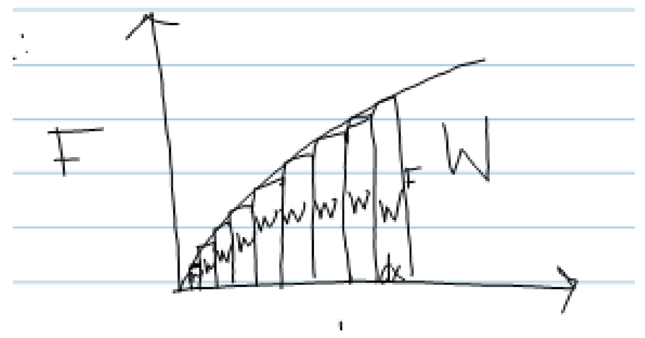
5. Representations

Students can depict layers of integration in various representations.

Diagrammatic:

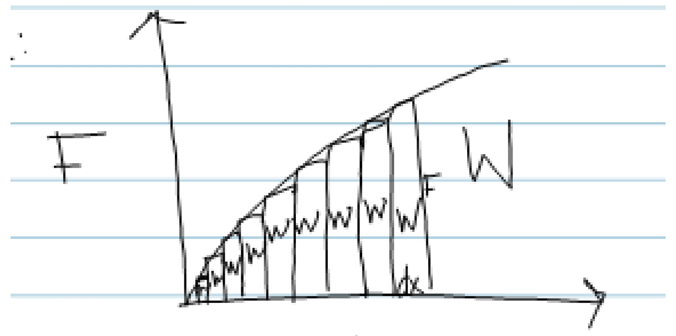


Graphical:

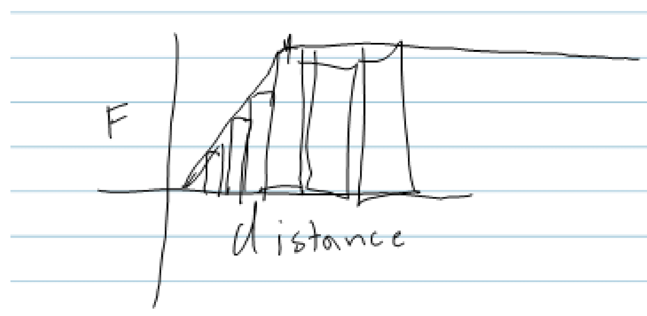


6. Assessment of Student Work

Two students may draw similar diagrams, but one may display more understanding of layers than another. For instance, this student-created diagram:



depicts work as a sum of small works, whereas this diagram:



portrays only a sequence of rectangles. The layers framework helps us to notice the important distinction between these two images.