FACILITATING STUDENTS' APPLICATION OF THE INTEGRAL AND THE AREA UNDER THE CURVE CONCEPTS IN PHYSICS PROBLEMS

by

DONG-HAI NGUYEN

B.S., Ho Chi Minh City University of Pedagogy, 2006

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Physics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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Abstract

This research project investigates the difficulties students encounter when solving physics problems involving the integral and the area under the curve concepts and the strategies to facilitate students learning to solve those types of problems. The research contexts of this project are calculus-based physics courses covering mechanics and electromagnetism.

In phase I of the project, individual teaching/learning interviews were conducted with 20 students in mechanics and 15 students from the same cohort in electromagnetism. The students were asked to solve problems on several topics of mechanics and electromagnetism. These problems involved calculating physical quantities (e.g. velocity, acceleration, work, electric field, electric resistance, electric current) by integrating or finding the area under the curve of functions of related quantities (e.g. position, velocity, force, charge density, resistivity, current density). Verbal hints were provided when students made an error or were unable to proceed. A total number of 140 one-hour interviews were conducted in this phase, which provided insights into students' difficulties when solving the problems involving the integral and the area under the curve concepts and the hints to help students overcome those difficulties.

In phase II of the project, tutorials were created to facilitate students' learning to solve physics problems involving the integral and the area under the curve concepts. Each tutorial consisted of a set of exercises and a protocol that incorporated the helpful hints to target the difficulties that students expressed in phase I of the project. Focus group learning interviews were conducted to test the effectiveness of the tutorials in comparison with standard learning materials (i.e. textbook problems and solutions). Overall results indicated that students learning with our tutorials outperformed students learning with standard materials in applying the integral and the area under the curve concepts to physics problems.

The results of this project provide broader and deeper insights into students' problem solving with the integral and the area under the curve concepts and suggest strategies to facilitate students' learning to apply these concepts to physics problems. This study also has significant implications for further research, curriculum development and instruction.

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Dedication

To my Mother, Nguyen Thi Ngoc Thu

Chapter 1 - Introduction

1.1 Motivation

Mathematics has often been considered the language of physics and other natural sciences for several centuries, as Galileo said: “the book of nature is written in the language of mathematics.” A well-established mathematics background is the foremost condition for understanding and communicating physics ideas. Mathematics also provides a useful toolbox for solving physics problems. For these reasons, most physics courses have mathematics pre-requisites. Despite this fact, students in introductory physics still struggle with applying their mathematics knowledge to physics problems. Significant research efforts have been devoted in diagnosing students’ difficulties in applying mathematics to physics and developing instructional strategies to help students overcome those difficulties. (J Tuminaro, 2004)

Integration is among the mathematical concepts that are widely used in physics. Many physics problems require calculating a physical quantity from other non-constant quantities using integration. Unlike mathematics problems in which the integrals are provided and the students only have to compute the integrals, most physics problems do not have pre-determined integrals. Instead, students have to set up an integral from the physics situation described in the problem statement and compute it. This process could be broken up into four steps:

- Step 1: recognize the need for an integral
- Step 2: set up the expression for the infinitesimal quantity
- Step 3: accumulate the infinitesimal quantities
- Step 4: compute the integral

An integral can be computed in several ways (e.g. by using integral techniques, evaluating a Riemann sum, calculating the area under the curve). The most common methods for calculating integrals in introductory physics problems are using integral techniques and calculating the area under the curve. Previous studies in physics education research have investigated students’ use and interpretation of the integral and the integral-area relation in physics (Cui, 2006; Manogue, Browne, Dray, & Edwards, 2006; L. C. McDermott, Rosenquist, & van Zee, 1986; Meredith & Marrongelle, 2008; Pollock, Thompson, & Mountcastle, 2007; Wallace & Chasteen, 2010). These studies investigate students’ conceptual understanding of the area under the curve (L. C. McDermott, et al., 1986), students’ application of the integral-area relation (Pollock, et al., 2007), the resources students use to cue integration (Meredith & Marrongelle, 2008), students' application and self-confidence in setting up the integrals in physics problems (Cui, Bennett, Fletcher, & Rebello, 2006), and students' difficulties in interpreting and calculating the integral in Ampere's law equation (Manogue, et al., 2006; Wallace & Chasteen, 2010). However, there is no study which investigates in detail the difficulties students encounter at each of the steps above, especially steps 2 and 3, and the difficulties students have with calculating an integral using the area under the curve. Moreover, despite the results of many studies that students have significant difficulties with integration in physics problem solving, there have been no instructional materials developed to facilitate students learning to apply integration in physics problems. As an effort to fill out these missing pieces of research, we conducted a research project which aimed at providing a complete description of the common difficulties students encounter at each of the steps above and creating tutorials to facilitate students' learning to solve physics problems involving integration.

1.2 Context of the research

The studies in this research project were conducted on students enrolled in the Engineering Physics course sequence at Kansas State University. This sequence consists of two courses of introductory calculus-based physics. At least one semester of calculus is the pre-requisite for enrolling in the Engineering Physics 1 course, which covers mechanics and thermodynamics. At least two semesters of calculus are the pre-requisites for enrolling in the Engineering Physics 2 course, which covers electricity and magnetism, geometric and physical optics. The problems used in this research project are in mechanics and electricity.

The courses are taught in the Studio format which includes two 50-minute lectures each week, and two two-hour long Studio sessions (Sorensen, Churukian, Maleki, & Zollman, 2006) in which students work on problems, go over homework and complete laboratory exercises. The Studios are facilitated by a primary instructor, who is typically an advanced graduate student, faculty member or post-doc and a secondary instructor, who is typically a beginning graduate student or undergraduate student.

1.3 Research questions

In phase I of the project, we investigate students' difficulties in applying the integral and the area under the curve concepts in physics problem solving. Specifically, we address the following research questions:

- To what extent did students recognize the use of the integral in physics problems?
- To what extent did students understand what quantity was being accumulated when calculating an integral?
- What were the common difficulties that students encountered when setting up and computing an integral algebraically in a physics problem?
- What verbal hints might help students overcome those difficulties?
- To what extent did students recognize the use of the area under the curve in physics problems?
- To what extent did students understand what quantity was being accumulated when calculating the area under the curve?
- To what extent did students understand the relationship between a definite integral and the area under a curve?

In phase II of the project, we created tutorials to facilitate students' learning to apply the integral and the area under the curve concepts in physics problems on work-energy and electricity. The research questions in this phase of the project are:

- To what extent did our tutorials on work-energy help students improve their ability to apply the integral and the area under the curve concepts in work – energy problems, compared to standard instruction (i.e. sample problems and solutions)?
- To what extent did our tutorials on electricity help students improve their ability to apply the integral and the area under the curve concepts in electricity problems, compared to standard instruction (i.e. sample problems and solutions)?

1.4 Research strategies overview

In phase I of the project, we investigated students' application of the integral and the area under the curve concepts in physics problem solving. We were interested in not only the difficulties students encountered but also the hints that might help students overcome those difficulties. So we conducted individual teaching/learning interviews with each of the 20 students recruited from the course. The teaching/learning interview format allows us to not only probe students' understanding but also facilitate students' learning. (Engelhardt, Corpuz, Ozimek, & Rebello, 2003; Steffe, 1983; Steffe & Thompson, 2000) The difficulties students encountered and the hints provided by the interviewer during the interviews were recorded, and were studied to find the emergent themes.

In phase II of the project, we created our tutorials based on the findings from phase I. The tutorials were implemented on students during the focus group learning interviews. In this interview, students worked in group of three to four on the worksheets provided by us. Students were asked to check in with the facilitator after they completed each exercise on the worksheet. The format of the focus group learning interviews, therefore, simulates the learning environment of a real recitation classroom. A pre-test and a post-test were implemented before and after the students received the treatments. Students' worksheets on the pre-test, post-test and the treatments were collected. The pre-tests and post-tests were then graded and statistical tests were employed to test the significance of the difference in scores between the groups.

1.5 Road map of dissertation

This dissertation consists of three major parts. The first part (including chapters 3 and 4) describes two studies in phase I of the project: the studies on students' difficulties with the integral and the area under the curve concepts in mechanics and electricity. The second part (including chapters 5 and 6) describes two studies in phase II of the project: the studies on the tutorials in mechanics and electricity. The third part (chapter 7) presents a pilot study on an attempt to use the transfer in pieces framework to track the development of students' application of the integral and the area under the curve concepts in mechanics problems.

In chapter 2, we review the previous research relevant to our studies. The relevant topics include students' difficulties with calculus concepts in mathematics and physics courses, transfer of learning, and tutorials in physics. In chapter 3, we discuss the methodology, rationale, and results of the study on students' application of the integral concept in mechanics and electricity problems. Chapter 4 discusses the same issues on students application of the area under the curve concept. In chapter 5, we describe the creation, implementation, and results of the tutorials on helping students learn to apply the integral and the area under the curve concepts in mechanics problems. Chapter 6 discusses the creation, implementation, and results of the tutorials on helping students learn to apply the integral concept in electricity problems. In chapter 7, we present two case studies in which we follow two students and they progress through our interviews in a semester period to trace the development of them in applying the integral and area under the curve. Chapter 8 summarizes the major results of the studies and discusses the implications for calculus and physics instruction as well as implications for further research on the topics.

Chapter 2 - Literature review

2.1 Chapter overview

In this chapter, we review the previous studies that are relevant to our research. Specifically, we review the literature on students' understanding and application of mathematical concepts in physics problem solving, the instructional materials that have been developed to facilitate this process, and the theoretical frame work we will use in analyzing students' application of mathematical concepts in physics problem solving.

Students participating in our interviews were recruited from several calculus-based physics courses. At least one or two semesters of calculus were the pre-requisites of these courses. So we start with a review of previous studies in mathematics education research on students' understanding of basic concepts. These studies might provide an idea on the common conceptions and misconceptions students hold when learning about calculus concepts. We first discuss students' understanding of basics concepts of calculus (e.g. function, limit, differentiation, integration) in calculus in sub-section 2.2.1 and then narrow down our discussion to students' understanding of the integral concept and the integral-area relation in calculus in sub-section 2.2.2.

Understanding a mathematical concept and being able to perform computation relevant to that concept are usually the criteria to measure a student's mastery of that concept. However, there have been many studies indicating that students are usually very fluent at computation while possessing very little conceptual understanding of the concepts underlying the computation. Obviously, this unbalance between conceptual knowledge and procedural knowledge will hurt students a lot when they have to use their mathematical knowledge in other disciplines, where the mathematical concepts are embedded in the a variety of contexts and students are not given pre-determined mathematical problem to solve. For example, most physics problems involving integral do not provide a pre-determined integral for students to compute. Instead, students have to set up the integral from the physical situation described in the problem statement. Students with little conceptual understanding about the integral might find task a very difficult or even impossible task. For this reason, we were interested in considering the state of the conceptual knowledge and procedural knowledge on basic calculus concepts that students passing calculus courses possess before they enter physics courses. This topic will be discussed in sub-section 2.2.3.

In section 2.3, we discuss previous studies in physics education research on how students use math in physics. We start with a review on studies that focus on students' application of mathematics in physics in general. Then we narrow down our discussion to the studies on students' understanding and application of the integral and the area under the curve concepts because these concepts are the focus of our study. We will briefly summarize each of the studies and relate it to our current study.

The literature on transfer of learning is discussed in section 2.4. Transfer of learning refers to the application of the knowledge learned in one context to other contexts, so it is closely related to our study which focused on the application of mathematical concepts in physical contexts. We will discuss both the traditional and contemporary models of transfer of learning, as well as a model that consolidate traditional and contemporary perspectives. We also introduce the transfer in pieces framework propose by Wagner (Wagner, 2006) which will be used for analyzing the students' application of the integral and the area under the curve in physics problems.

One of the objectives of our research project was to develop tutorials to facilitate students' learning to apply the integral and the area under the curve concepts in physics problem solving. So in section 2.5, we discuss some of the tutorials that have been developed to facilitate students' learning in introductory physics. These tutorials focus on improving students' conceptual understanding of physics concepts by providing more opportunity for students to explore the concepts and resolve the conflicts between their intuitive models and the Newtonian models of the concepts. Our tutorials, instead, aimed at improving students' mathematical skills in physics problem solving. The chapter concludes with a summary of the literature discussed in the sections.

2.2 Research in mathematics education

2.2.1 Students' understanding of basics concepts of calculus

There have been several studies in mathematics education research on students' understanding of basics concepts of calculus. Among the earliest research was the work of Orton

(Orton, 1983). In that study, 110 British students aged 16 – 22 were interviewed on several tasks involving the concepts of limit and integration. Many of these tasks involved finding the area under a curve using the Riemann sum method and calculating the limit of that Riemann sum. Some other tasks asked students to prove basic properties of integration (such as the integral of a sum was the sum of integrals) using area under the curve. Orton found that students' errors with these basic concepts of calculus could be classified as structural (fundamental or conceptual), executive (operational and procedural), or arbitrary. Structural errors were the errors "which arose from some failure to appreciate the relationships involved in the problem or to grasp some principle essential to solution." Executive errors occurred when students failed to carry out the calculations although they might have understood the principle involved in the problem. Arbitrary errors were made when students followed arbitrary strategy for solving the problem that violated the constraints set up by the given information in the problem. For example, in a problem in Orton's study, the students were asked to evaluate the area under the curve of $y = x^2$ from $x = 0$ to $x = a$ using the staircase method (i.e. approximate the area under the curve as a sum of the areas of several rectangles). The problem was divided into three questions. The first question asked students to calculate the width of each rectangle if the area was divided into six rectangles. The second question asked students to calculate the heights of each rectangle. The third question asked students to calculate the total area under the curve. Thirteen students made errors in this first task, but eventually were able to obtain $a/6$ for the width. Three students were then unable to find the height of the rectangles. These three students were said to make structural error because they did not recognize that the relationship between the width and the height of the rectangles was also the relationship between the x-values and the y-values on the curve. The other 10 students were able to recognize this relationship but then failed to get the correct height of the rectangles because of the errors in computation. These errors were executive errors.

Orton also found that the majority of students did not view the integral as the limit of a Riemann sum and talked about such limit as an approximation, not as an exact answer, although they had no difficulty evaluating a given Riemann sum. Investigating students' conceptual versus procedural abilities, Orton found that most of the students in his study were able to carry out the procedures and techniques of integration although they might not have good understanding of the underlying concepts.

Ferrini-Mundy and Graham (Ferrini-Mundy & Graham, 1994) interviewed a group of six students in calculus to reveal students' understanding of basic concepts of calculus (e.g. function, limit, continuity, derivative, and integral) and the interrelationships among those concepts. They investigated in details the performance of one student – Sandy – in the interviews. They found that Sandy and many other students in the study "interpreted the integral as a signal to 'do something'." She perceived the definite integral as "the area between the graph of the function and the x-axis." while thinking of the sum of the areas of the small rectangles under the curve as the "proof" for that fact.

2.2.2 Students' understanding of the integral concept and the integral-area relation

Rasslan and Tall (Rasslan & Tall, 2002) investigated the definition and images of the definite integral held by high school students in UK. They found that "the majority do not write meaningfully about the definition of definite integral, and have difficulty interpreting problems calculating areas and definite integrals in wider contexts." They suggested strategies for teaching the definite integral concept. The strategy was to introduce the concept as "cases extended the students' previous experience" and let the students experience it in use through a variety of examples covering a wide contextual range.

Sealey (Sealey, 2006) investigated students' problem solving on "real world problems" involving integration in a calculus class. The "real world problems" in this study were physics problems in which physical quantities were calculated using integration. She found that students might be proficient in dealing with the area under a curve but they might not be able to relate such an area to the structure of a Riemann sum. She concluded that the area under the curve method could be a powerful tool to evaluate a definite integral only when students understood the structure of the definite integral.

Also emphasizing on the importance of understanding the structure of the definite integral, Thompson and Silverman (Thompson & Silverman, 2007) pointed out that for students to perceive the area under a curve as representing a quantity other than area (e.g. velocity, work), it was important that students considered the quantity being accumulated as a sum of infinitesimal bits that were formed multiplicatively. Thompson and Silverman proposed the accumulation model which considered integration as an accumulation of the bits that were made of two multiplicative quantities. This model emphasized the two "layers" of integration: the multiplicative layer when the bits were formed and the accumulative layer when the bits were accumulated.

In our study, we found evidence of students' failure in interpreting the meaning of the area under the curve when they did not perceive it as a Riemann sum and did not understand the structure of the Riemann sum. We also used the structure of the Riemann sum as hints to help students set up the correct integral or recognize the meaning of the area under the curve.

2.2.3 Students' procedural knowledge and conceptual knowledge in calculus

As mentioned in the previous section, students are usually very fluent in computing a mathematical task while having very little conceptual understanding of the concepts underlying that computation. This unbalance between conceptual and procedural knowledge will lead to students' difficulties in applying their mathematical knowledge to physics, which will be revealed in our study.

There have been a number of research studies in mathematics education on students' conceptual versus procedural knowledge on basic concepts of calculus (Artigue, 1991; Chapell & Killpatrick, 2003; Engelbrecht, Harding, & Potgieter, 2005; Hiebert & Lefevre, 1986; Mahir, 2008). A student is said to have conceptual knowledge if the student possesses the knowledge and recognizes its connection to other pieces of knowledge. In this sense, the connection between the pieces of knowledge is as important as the knowledge itself. Procedural knowledge refers to the rules, algorithms, and techniques that are used to solve mathematics problems. (Hiebert & Lefevre, 1986) Artigue investigated calculus students' understanding of differentiation and integration. He found that although most of the students could perform routine procedures for finding the area under a curve, rarely could they explain their procedures. Some students did not even realize why they were doing it. (Artigue, 1991) This finding is shared by the study of Oaks (1987). He found that "some students are not even aware that there are concepts underlying the procedures they use." (Oaks, 1987)

Mahir (2008) investigated the conceptual and procedural knowledge of 62 students who had successfully completed a one-year calculus course. These students were asked to solve five calculus problems relating the concepts of integral, integral – are relation, integral as a sum of areas, and the fundamental theorem of calculus. The first two problems (1 and 2) could be solved using integral formulas and techniques, so these problems could evaluate students' procedural knowledge. The next two problems (3 and 4) could be solved by using either the integral – area relation or integral techniques. The last problem (problem 5) was more complicated and required the students to combine many concepts, so it served to evaluate students' conceptual knowledge. Mahir found that the majority of the students successfully solved the first two problems (92% on problem 1 and 74% on problem 2) while there was only 24% of the students succeeded in problem 5 and 40% of the students did not respond to this problem. On problems 3 and 4, the majority of the students followed the procedural approach and there was only a small portion of the students followed the conceptual approach. The percentages of students following the conceptual approach and obtained the correct answers were 100% on problem 3 and 71% on problem 4, while these percentages of the students who followed the procedural approach were 11% and 16% on corresponding problems. Mahir concluded that the students in his study did not have satisfactory conceptual understanding of the concepts being tested. He also concluded that the students following the conceptual approach also performed satisfactorily on procedural calculations and had a higher success rate than the students following the procedural approach. He suggested that concept-based instruction might help to improve students' conceptual understanding in calculus. This suggestion was supported by the study of Chapell and Killpatrick which found that "the students exposed to the concept-based learning environment scored significantly higher than the students in procedural-based environment on assessment that measures conceptual understanding as well as procedural skills." (Chapell & Killpatrick, 2003)

Students' inclination to use procedural knowledge rather than conceptual knowledge might be explained by their reluctance to visualize the mathematics problems. Eisenberg and Dreyfus (1985) pointed out that "students had a strong tendency to think algebraically rather than visually … even if they are explicitly and forcefully pushed towards visual processing." This finding was supported by the studies other researchers (Monk, 1988; Mundy, 1984; Swan, 1988; Vinner, 1989). One reason for this was pointed out by Eisenberg and Dreyfus (1991): there is a common belief among mathematicians, teachers, and students that "mathematics is non-visual, regardless of whether or not a visual representation is at the base of an idea." (p. 30) This reason was also mentioned in the study of Aspinwell and Miller (Aspinwell & Miller, 1997) for the case of calculus: "students regard computation as the essential outcome of calculus and thus end their study of calculus with little conceptual understanding."

2.3 Research in physics education

2.3.1 Students' difficulties with mathematics in physics

Research in physics education indicates that students encounter significant difficulties when applying their mathematical knowledge and skills to physics problem solving. The major cause of these difficulties is not the students' lack of the required mathematical knowledge to solve the problems, but their inability to apply their mathematical knowledge to physics problems. (J. Tuminaro & Redish, 2004) Even when students are able to apply a particular mathematical concept to a physics problem, they might not conceptually understand the mathematical processes although they can easily carry out the calculations. (L. C. McDermott, 2001)

Yeatts and Hundhausen (Yeatts & Hundhausen, 1992), based on their teaching experience, classified students' difficulties when transferring from calculus to physics in three categories. The first category - "notation and symbolism" - included difficulties that arose from students' rote memory of, and hence, reliance on the symbols used in each context. Mathematics and physics might use the same notation or symbol to mean different things, thus causing difficulties to students. The second category of difficulty - "the distraction factor" - occurred when the surface features of the problem prevented students from seeing the underlying mathematical process in a physics problem. The third category was "compartmentalization of knowledge," which occurred when students stored knowledge of different disciplines in different "cabinets" and activated knowledge in each "cabinet" only in the corresponding discipline.

Tuminaro (2004) investigated the reasons for students' poor performance on mathematical related tasks in physics problem solving and the strategy for improving the situation. He provided evidence that the major cause for students' failure in applying mathematical knowledge to physics problem solving was not the lack of the necessary knowledge but the inability to apply that knowledge in a physics context. He proposed a theoretical framework for analyzing students' application of mathematics in physics. This framework identifies three levels of cognitive structures relevant to mathematical thinking and physics problem solving: mathematical resources, epistemic games, and frames. Using his framework, Tuminaro cited a number of reasons for students' failure to apply mathematics to physics. These included using an inappropriate recourse, using an appropriate resource but mapping it to physics context inappropriately, playing the appropriate epistemic game but making wrong moves within the game, playing the inappropriate epistemic game due to incorrect framing of the problem. He suggested reframing as an effective instructional strategy that might help students activate different resources and epistemic games to solve physics problems.

Bing and Redish (Bing & Redish, 2008) proposed a model for the association of knowledge in mathematics and physics to make sense of a physics equation or idea. They called this association “the cognitive blending of mathematics and physics knowledge.” They adopted two types of blending described by Fauconnier and Turner (Fauconnier & Mark, 1998) to investigate the combination of mathematical and physical knowledge and reasoning. These two types of blending are called single-scope and double-scope blends. A single-scope blend imports knowledge elements from one input mental space (i.e. the knowledge elements and patterns that one has on a specific topic) into the organizing frame of the other mental space. A double-scope blend “displays a blending of the organizing frames of the input mental spaces.” (Bing & Redish, 2008) The cognitive blending framework “emphasizes the demands students face concerning the integration of their mathematical and physical knowledge.” and may help instructors understand students’ thinking and provide scaffolding to prompt students to blend their knowledge in a productive way for the situation at hand. Facilitating students to blend their mathematical and physical knowledge is also the ultimate goal of the tutorials we developed in our study.

2.3.2 Students’ application of the integral concept in physics

Cui et al. (2006) investigated students’ retention and transfer from calculus to physics. They found that students had significant difficulties distinguishing variables and constants in an integral as well as determining the limits of an integral. They also found that four out of seven interviewees recognized the use of integral in a physics problem by recalling the strategy they had learned from in-class examples while the other three students had a rough idea of an integral as a sum of an infinite number of small elements.

Meredith and Marrongelle (2008) investigated the resources that students used to cue integration in electrostatics problems. They used the notion of Sherin’s symbolic forms to describe these resources. (Sherin, 2001) A symbolic form is a cognitive mathematical primitive which allows students to “associate a simple conceptual schema with an arrangement of symbols in an equation.” (p. 482) Meredith and Marrongelle identified three symbolic forms that students used to cue integration, namely, the recall cue, the dependence cue, and the parts-of-a-whole cue. Recall is not a symbolic form because it does not have a mathematical structure, but it is commonly used in cueing integration. The recall cue is identified when students recall a previously learned strategy when solving a problem. The dependence symbolic form is described as “a whole depends on the quantity associated with an individual symbol.” The dependence cue is identified when students decide to integrate because there is a quantity that depends on another quantity. The parts-of-a-whole symbolic form is described as “amounts of generic substance, associated with terms that contribute to a whole.” Interpreting an integral as an accumulation of infinitesimally small elements indicates the use of parts-of-a-whole cue. Meredith and Marrongelle also found that the dependence cue was more commonly used by students than the parts-of-a-whole cue, although “the use of the dependence symbolic form led to inaccuracies if the quantity being integrated was not a rate or a density.” (p. 577) They suggested that the parts-of-a-whole symbolic form was a more powerful and flexible resource to cue integration and proposed instructional strategies to promote students’ use of the parts-of-a-whole resource to cue integration in physics problems.

Most recently, Wallace and Chasteen (2010) found that part of students’ difficulties with Ampère’s law was due to students not viewing the integral in Ampère’s law as representing a sum, which aligned with the work of Manogue et al. (2008) on the same topic.

In our point of view, the application of integration in a physics problem can be divided into four steps:

- Step 1: recognize the need for an integral
- Step 2: set up the expression for the infinitesimal quantity
- Step 3: accumulate the infinitesimal quantities
- Step 4: compute the integral

The work by Meredith and Marrongelle (2008) investigated the first step. Although they did mention that students might misapply the symbolic forms in setting up an integral, they did not investigate this misapplication in details. The work of Cui *et al.* (2006) mentioned some of the difficulties students had when applying integral in physics (i.e. step 2) but did not discuss them in any significant detail. Our current study adds the missing piece to the picture. We investigate students’ difficulties in all four steps of the process, especially those in steps 2 and 3, which have previously not been discussed in detail in the literature

2.3.3 Students' application of the area under the curve concept in physics

There have been a few studies in physics education research that focus on how students apply the area under a curve method in evaluating integrals in physics problems. McDermott et al. (1986) investigated students' difficulties in connecting graphs and physics in the context of kinematics. They identified two categories of difficulty students had with graphs. First, students had difficulties in connecting graphs to physics concepts, including discriminating between the slope and height of a graph, interpreting changes in height and changes in slope, relating one type of graph to another, matching narrative information with relevant features of a graph, and interpreting the area under a graph. Second, students had difficulties in connecting graphs to the real world, including representing continuous motion by a continuous line, separating the shape of a graph from the path of the motion, representing a negative velocity on a "v vs. t" graph, representing constant acceleration on a "a vs. t" graph, and distinguishing among different types of motion graphs.

In a problem involving finding displacement from a graph of "v vs. t", students had to find the area under the curve by counting the number of squares bounded by the curve and the $v = 0$ axis and then multiplied it by the displacement that each square represented. They found that most of the difficulties students had were directly related to their "inability to visualize the motion depicted by the velocity versus time graph." (p. 506) Students did not know which square they should include in the "area under the curve," so they counted all of the squares from under the curve all the way to the bottom line of the grid where the horizontal axis was labeled. That led to students' difficulties in distinguishing positive and negative areas, as well as associating them with displacement in positive and negative direction respectively.

More recently, Pollock et al. (2007) investigated students' understanding of the physics and mathematics of process variables in P-V diagrams in thermodynamics. On a question asking students to compare the work done by a gas taking two different paths on the P-V diagram, they found that successful students were those who recognized that work was $\int PdV$ and that this integral equaled the area under the path.

2.4 Transfer of learning

Transfer of learning is defined as the ability to apply the knowledge one has learned in one situation to another situation. (Reed, 1993; Singley & Anderson, 1989) Transfer of learning during problem solving involves the application of the knowledge and problem solving technique one has learned in a particular problem to another problem. In physics problem solving, the instructor usually solves a sample problem to demonstrate how a physical principle applies in a particular or a general problem. Students are then expected to solve other problems involving that principle, i.e. to transfer the problem solving technique they learn from the sample problem to other problems.

2.4.1 Traditional models of transfer

The traditional models of transfer (Adams et al., 1988; Bassok, 1990; Brown & Kane, 1988; Chen & Daehler, 1989; Nisbett, Fong, Lehmann, & Cheng, 1987; Novick, 1988; Perfetto, 1983; Reed, Ernst, & Banerji, 1974; Throndike & Woodworth, 1901; Wertheimer, 1959) considered transfer from the researcher's perspective. In this perspective, transfer is a passive, static process in which students apply the knowledge they learn in one situation to another situation. Transfer, therefore, depends on how similar the learning situation is to the transfer situation. According to Thorndike's theory of identical elements, transfer from one activity to another occurs only if the activities share common surface features. (Throndike, 1906) On the other hand, Judd's theory of deep structure transfer suggest that transfer depends on how much of the underlying principles (i.e. deep structure) are noticed by the learner. (Judd, 1908) Despite the difference in what causes transfer to happen, these two theories share a common point: the knowledge to be transferred between situations has been pre-defined by the researcher.

2.4.2 Contemporary models of transfer

Transfer researchers have changed their perspective as they recognized a severe lack of evidence supporting the previous models of transfer. (N. Sanjay Rebello, 2007) Contemporary models of transfer consider transfer from the learner's perspective. In this perspective, transfer is an active, dynamic process in which the learner constructs a new knowledge structure in the new situation. These models focus not only on the cognitive aspect but also on the socio-cultural aspect of transfer.

Lobato actor-oriented transfer (AOT) model (Lobato, 2003) conceives transfer as the personal construction of similarities between activities. In this model, the researcher does not define the knowledge which the learner is expected to transfer. Instead, the knowledge to be transferred depends on what the learner perceives as similar between the situations. In this sense, the AOT model examines transfer from the learner's point of view rather than the researcher's point of view. The role of the research is to find out what students transfer and investigate the mediating factors.

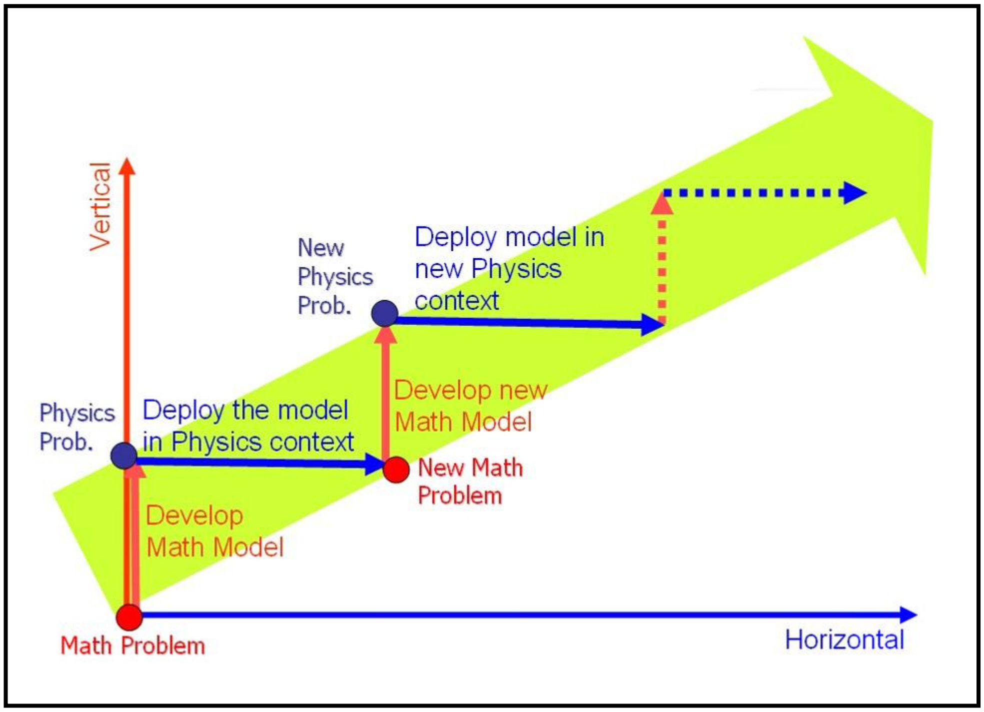
Bransford and Schwartz (Bransford & Schwartz, 1999) consider transfer from the preparation for future learning point of view. They are interested in whether the learners can learn to solve problems in the new contexts. They believe that transfer is likely to occur if the learners reconstruct their learning in the new context in the same way as they did in the learning context.

2.4.3 Consolidating traditional and contemporary models of transfer

Based upon Redish's two-level framework (Redish, 2004), Rebello (N. S. Rebello, Cui, Bennett, Zollman, & Ozimek, 2007) developed a new framework that consolidates both the traditional and contemporary perspectives about transfer in such a way that both of these two types of transfer are valued and promoted in learning. This framework considers transfer as the dynamic creation of associations between prior knowledge and read-out information from a given situation. According to this framework, there are two kinds of associations that a learner can make. The first kind of association occurs when a learner assigns information read out from the situation to an element in his or her own prior knowledge. The second kind of association occurs when the learner establishes a link between the readout information and an element of their prior knowledge structure.

These associations are related to two different transfer processes: horizontal and vertical transfer. Horizontal transfer involves the application of a well developed knowledge to new situations. A learner possesses a well developed schema for solving a problem which is invoked when the problem is encountered. The learner 'plugs-in' information from the problem into the schema. An example of horizontal transfer occurs when solving a simple 'plug-n-chug' problem. Vertical transfer occurs when a learner encounters a problem that cannot be solved using an existing schema. Then they must adapt and reconstruct their schema to incorporate new knowledge to solve the problem. Scaffolding is often needed to facilitate vertical transfer. The ability of a learner to creatively adapt to a new problem is called the adaptive expertise. (Schwartz, Bransford, & Sears, 2005) To gain adaptive expertise a learner must navigate a sequence learning experiences involving vertical and horizontal transfer as presented in Figure 2.1 below.

Figure 2.1 Theoretical framework showing sequences of vertical and horizontal transfer needed to achieve adaptive expertise



The tutorials we developed to facilitate students' application of the integral and the area under the curve concepts in work-energy problems in this study followed the horizontal and vertical aspects of transfer. The sequence of matched math and physics exercises in the tutorial was intended to facilitate both horizontal and vertical transfer in problem solving. The math exercises provided the opportunities to develop representational models of the mathematical concept, so they involve vertical transfer. The physics exercises provided the opportunity to apply these models in physics contexts, so they involve horizontal transfer.

2.4.4 The transfer in pieces framework

In our study, we also employ the transfer in pieces framework proposed by Wagner (Wagner, 2006) to interpret and trace the development of students' application of the integral and the area under the curve concepts in physics problem solving. This framework was developed based on diSessa's knowledge-in-pieces epistemology. (diSessa, 1993) In diSessa's framework, knowledge consists of fine-grained pieces called phenomenological primitives (p-prims). A p-prim is an elementary piece of knowledge which is self-explanatory and is neither right nor wrong. Students may explain or interpret a physical phenomenon by activating different p-prims. Different aspects of the same phenomenon might be interpreted by different combinations of p-prims. For students who perceive physics as a collection of facts, their physics p-prims are usually not organized and the relation between the p-prims is not attended. Therefore, these students might employ very different p-prims to explain phenomena having different contextual features although they share common physical principles.

diSessa also proposed a model for a particular type of concept – a coordination class – which is "systematically connected ways of getting information from the world." (diSessa & Sherin, 1998) A coordination class has two major structural components: the readout strategies and the causal net. The readout strategies determine how the characteristic attributes of a concept are seen or "read out" from a given situation. The causal net is a class of knowledge and reasoning strategies that determine when and how an observation is related to the desired information. Different readout and coordination strategies might be required to perceive and interpret the same concept embedded in different contexts.

Wagner introduced the term concept projection which is "a specific combination of knowledge resources and cognitive strategies used by an individual to identify and make use of a concept under particular contextual conditions." (Wagner, 2006) These contextual conditions may be any surface features of the problem such as representation or the cover story in which the concept is embedded. So an individual's understanding of a particular concept might be supported by several different concept projections corresponding to different situations in which the concept is applicable. These different concept projections might share some common knowledge resources but also contain their own knowledge resources that make them applicable in certain situations but not in others. The span of a concept projection is defined as the range of contexts across which that concept projection is found to be applicable (Wagner, 2006). According to the concept projection framework, "transfer is understood not as the all-or-nothing transportation of an abstract knowledge structure across situations, but as the incremental growth, systematization, and organization of knowledge resources that only gradually extend the span of situations in which a concept is perceived as applicable." (Wagner, 2006, p. 10) This framework emphasizes the difference between having a concept and being able to apply that concept in different situations.

In chapter 7 of this dissertation, we will analyze the performance of two students over a semester on several problems involving the integral and the area under the curve concepts in the light of the transfer in pieces framework. We will interpret students' application of these two concepts in terms of the concept projections that students build for them and how those concept projections are related to students' success or failure on the tasks.

2.5 Tutorials to facilitate students' learning in introductory physics

As students' misconceptions about basic physics concepts are revealed by many researchers, effort has been put into developing instructional materials to help students correct those misconceptions. These instructional materials are usually in form of tutorials, which are worksheets carefully designed for students to work in small groups on activities in which students' intuitive knowledge about a physical principle is challenged and is eventually replaced by formal knowledge. The students are usually the major subjects in this process. The instructor only acts as a facilitator to help students work through the process by asking questions to probe students' understanding of the topic, reveal their misconceptions, and provide scaffolding if necessary. We will discuss some of the well-know tutorials in physics below.

The *Tutorial in Introductory Physics* (TIP) is a well-known instructional material developed by the Physics Education Group at the University of Washington. (L. C. McDermott & Shaffer, 1998) The TIP addresses students' misconceptions through a three-step process: elicit, confront, and resolve. Students are first presented with a situation in which students' misconceptions are found to come into play. Once the misconception has been brought up, students are asked questions that might lead to cognitive conflicts between students' intuitive knowledge and the actual situation at hand. As the students reconsider their ideas about the situation, the tutorial provides scaffolding to help students build formal knowledge about the situation.

The Physics Education Research Group at the University of Maryland has also developed tutorials for introductory physics, which is known as the *Activity-based Tutorials* (ABT). (Wittmann, Steinberg, & Redish, 2004) The ABT makes use of hands-on experiments and microcomputer-based data acquisition techniques with which students explore the principles of physics themselves. This group is also the authors of the *Open-Source Tutorial* (OST) which treats students' intuitions about the world as valuable observations that might have been misinterpreted. (Scherr & Elby, 2007) The OST then helps students recognize that their intuitions only apply to a limited number of situations or certain aspects of the situation at hand, and modify these intuitions so that they cover a broader range of situations or take into account all aspect of the situation at hand. In other words, the OST helps students refine their "raw intuitions" about the physical world.

These tutorials have been proven to have positive impacts on students' conceptual understanding in physics compared to traditional instruction. (L. C. McDermott, 2001; L. C. McDermott & Redish, 1999) Smith and Wittmann (Smith & Wittmann, 2007) compared the effectiveness of the three tutorials mentioned above in helping students understand Newton's third law in an algebra-based physics course. Each of the tutorials was implemented to one-third of the students in the same course during the regular tutorial sessions of the course. Students' understanding of Newton's third law after the lecture, before and after the tutorial, in course examinations, and on the Force and Motion Conceptual Evaluation (FMCE) was investigated. They found that all three tutorials improved students' understanding of Newton's third law, but the OST was more effective than the other two tutorials.

All of the tutorials mentioned above use guided-inquiry method to improve students' conceptual understanding of basic concepts in introductory physics and minimize mathematical problem solving. Nevertheless, problem solving is an important aspect of learning introductory physics. Many researchers have reported on the difficulties students encounter when solving physics problems, especially their poor performance on mathematical tasks in physics problems, as discussed in section 2.2 of this chapter. Mathematics is an important tool in physics and being able to apply mathematics knowledge and skills to physics problems is one of the most crucial goals of physics instruction. In our study, we develop tutorials that aim at helping students learn to apply the integral and the area under the curve concepts to physics problems. Our tutorials focus not only on improving students' conceptual understanding of the mathematical and physical concepts, but also on improving students' ability to apply the mathematical concepts in physical contexts.

2.6 Chapter summary

In this chapter, we have reviewed the literature related to our research. We discuss four main topics:

- Students' understanding of the basic concepts in calculus
- Students' application of mathematics in physics problem solving
- Transfer of learning from both traditional and contemporary perspectives
- Tutorials to help students learn physical concepts.

Research in mathematics education has shown that students did not have satisfactory conceptual understanding of basics calculus concepts even though they could perform the calculations fluently. Students also expressed a strong preference on algebraic method to graphical (visualized) method in solving calculus problems.

Research in physics education have indicated that students had significant difficulties applying mathematics to physics. This is not due to the lack of the necessary mathematics resources but due to students' inability to activate those resources in physics contexts. Even when students are able to activate a mathematics resource and carry out the calculation, there is evidence that students do not understand the process underlying that calculation. Students' difficulties in applying the integral and the area under the curve concepts in physics problems are also reported.

We examine some of the tutorials developed to enhance students' understanding of physics concepts. Among the most well-known tutorials are the Tutorials in Introductory Physics (TIP) by University of Washington, the Activity-Based Tutorials (ABT) and the Open-Source Tutorial (OST) by University of Maryland. These tutorials have been reported as more effective than traditional instruction in helping students' learn basic concepts of physics. The OST is shown to be the most effective in teaching students about Newton's third law. These tutorials aim at improving students' conceptual understanding and minimize problem solving. The tutorials we create in our study also aim at helping students improve their understanding of mathematical and physical concepts, but more importantly, our tutorials aim at facilitating students' application of the integral and the area under the curve concepts in physics problem solving.

We also discuss transfer of learning from both the traditional and contemporary perspectives, particularly the vertical and horizontal aspects of transfer. We briefly describe the components of our tutorials and how they fit into the vertical versus horizontal transfer framework. We also describe the transfer in pieces framework, which we will use to trace students’ conceptual development throughout our interviews.

Chapter 3 - Students' application of the integral concept in physics problems

3.1 Introduction

In phase 1 of the project, we investigated students' performance on physics problems in which parts of the information were given as mathematical functions in algebraic and graphical representations. In this project, a function was considered to be in algebraic representation if it was defined by an equation, for example, $f(x) = 2x^2 - 3x + 1$. A function was considered to be in graphical representation if it was defined by the graph of that function with respect to its variable.

We conducted individual teaching – learning interviews with several students on a variety of problems in mechanics and electricity and magnetism. In each of these interviews, the students were asked to solve two isomorphic problems: an algebraic problem (i.e. a problem involving the algebraic representation of a function) and a graphical problem (i.e. a problem involving the graphical representation of a function). In the algebraic problem, students had to calculate a physical quantity by setting up and computing an integral algebraically (i.e. performing the integration). In the graphical problem, the integral must be computed graphically by evaluating the area under the curve. We investigated the difficulties students encountered when they set up and computed the integrals in these problems and the hints that might help students overcome those difficulties.

In this chapter, we will investigate students' performance on the algebraic problems. Specifically, we look at how students set up an integral representing a physical quantity from the problem statement and how they computed that integral algebraically.

The following research questions will be addressed in this chapter:

RQ1: Did students recognize the use of the integral in physics problems?

RQ2: Did students understand what quantity was being accumulated when calculating an integral?

RQ3: What were the common difficulties that students encountered when setting up and computing an integral algebraically in a physics problem?

RQ4: What verbal hints might help students overcome those difficulties?

3.2 Methodology

3.2.1 The individual teaching – learning interview

In the spring semester of 2009, 20 students were selected from 102 volunteers enrolled in a first semester calculus-based physics (Engineering Physics 1 or EP1). Students present at the first lecture of the course were offered the opportunity to volunteer for this study. Students were selected based on how their availability matched with that of the interviewer. The selected students were given monetary incentive for their participation. They were paid $40 for their participation in a series of four interviews, each an hour long. Most of these students were freshmen or sophomores in engineering majors and had taken physics in high school. Three of them were international students. Among these 20 students, there were 13 males and 7 females.

Table 3.1 Demographics of student participants in the spring 2009 interviews

Code ID	Year	Major	Semester	Physics Background
S1	1	Mechanical Engineering	Spring 09	High School Physics
S2	1	Mechanical Engineering	Spring 09	High School Physics
S3	2	Architectural Engineering	Spring 09	High School Physics
S4	1	Chemical Engineering	Spring 09	High School Physics
S5	2	Chemistry	Spring 09	None
S6	1	Electrical Engineering	Spring 09	High School Physics
S7	1	Electrical Engineering	Spring 09	High School Physics
S8	2	Electrical Engineering	Spring 09	High School Physics
S9	1	Mechanical Engineering	Spring 09	High School Physics
S10	1	Mechanical Engineering	Spring 09	High School Physics
S11	1	Chemical Engineering	Spring 09	High School Physics
S12	2	Civil Engineering	Spring 09	High School Physics
S13	1	Environmental Engineering	Spring 09	None
S14	1	Mechanical Engineering	Spring 09	High School Physics

S15	1	Chemical Engineering	Spring 09	High School Physics
S16	1	Electrical Engineering	Spring 09	High School Physics
S17	1	Mechanical Engineering	Spring 09	High School Physics
S18	1	Open Option	Spring 09	High School Physics
S19	1	Electrical Engineering	Spring 09	High School Physics
S20	1	Civil Engineering	Spring 09	High School Physics

Each of these 20 students was scheduled for four one-hour individual interviews during the spring 2009 semester. We will label these interviews as interviews 1 through 4. Each interview occurred within two weeks after an exam in the EP1 course. The topics of the interview problems were those had been tested in the most recent exam.

- Interview 1: One-dimensional kinematics
- Interview 2: Work and energy without friction
- Interview 3: Work and energy with friction
- Interview 4: Work and energy in rotational motion

In each interview, a student was asked to solve three problems:

- Original problem: A problem from the most recent exam. This problem was a typical end-of-chapter problem, and was given to help students get familiar with the physics principles covered in the interview.
- Graphical problem: A modified version of the original problem in which part of the information was given as a graph of a function.
- Algebraic problem: A modified version of the original problem in which part of the information was given as an algebraic expression of a function.

In order to investigate the effect of the problem sequence on students' performance, in each of the interviews 2 through 4, approximately half of the students were given the algebraic problem before the graphical problem (which we called the A-G sequence), and the other half of the students were given the graphical problem before the algebraic problem (which we called the G-A sequence). The number of students following each sequence is presented in Table 3.2.

Table 3.2 Number of students following each sequence in the spring 2009 interviews

Interview	A-G sequence	G-A sequence	Total
2	9	11	20
3	11	9	20
4	11	9	20

In the fall semester of 2009, fifteen students from the spring 2009 interviews, who were enrolled in a second-semester calculus-based physics course (Engineering Physics 2 or EP2) at that time, agreed to continue with our study in electricity and magnetism. Among these 15 students, there were 9 males and 6 females. Table 3.3 below provides some basic demographic information about the students participating in the fall 2009 interviews.

Table 3.3 Demographics of student participants in the spring 2009 interviews

Code ID	Year	Major	Semester	Physics Background
S1	1	Mechanical Engineering	Fall 2009	High School Physics
S2	1	Mechanical Engineering	Fall 2009	High School Physics
S3	2	Architectural Engineering	Fall 2009	High School Physics
S4	1	Chemical Engineering	Fall 2009	High School Physics
S5	2	Chemistry	Fall 2009	None
S6	1	Electrical Engineering	Fall 2009	High School Physics
S7	1	Electrical Engineering	Fall 2009	High School Physics
S8	2	Electrical Engineering	Fall 2009	High School Physics
S9	1	Mechanical Engineering	Fall 2009	High School Physics
S10	1	Mechanical Engineering	Fall 2009	High School Physics
S11	1	Chemical Engineering	Fall 2009	High School Physics
S12	2	Civil Engineering	Fall 2009	High School Physics
S13	1	Environmental Engineering	Fall 2009	None
S14	1	Mechanical Engineering	Fall 2009	High School Physics
S15	1	Chemical Engineering	Fall 2009	High School Physics

Each of these students went through another sequence of four interviews (interviews 5 through 8) during the fall 2009 semester. The format of these interviews was similar to that of the spring 2009 interviews, except that there were four or five problems in each interview. These problems included one problem with constant quantities and other problems with non-constant quantities, the information of which was provided as mathematical functions in algebraic and graphical representations. Each graphical problem in the fall 2009 interviews contained three to four graphs of related quantities. In this semester, we did not investigate the effect of the problem sequence on students' performance, so all students were given the problems in the same sequence (the A-G sequence) in all interviews. The topics of each interview were:

- Interview 5: Charge distribution and electric field
- Interview 6: Resistance and capacitance
- Interview 7: Current density and Ampere's law
- Interview 8: RLC circuit at resonance

In all interviews in both the spring and fall 2009 semesters, students were asked to think aloud as they solved the interview problems. Verbal hints were given by the interviewer whenever students made an error or were unable to proceed. All students were able to obtain the correct answers for all problems within the one-hour limit of each interview. All interviews were video-taped and audio-taped and were fully transcribed. Students' worksheets as well as the interviewer's field notes were also collected.

Table 3.4 The similarities and differences between the spring and fall 2009 interviews

Semester	Spring 2009 interviews	Fall 2009 interviews
Similarities	- Individual teaching/learning interviews - Same cohort of students - Problems with algebraic and graphical representations of functions - Students think aloud as they solve the problems - Verbal hints are provided when students make an error or are unable to proceed	
Differences	- 20 students	- 15 students

	- 3 problems in each interview - A-G sequence vs. G-A sequence	- 4 or 5 problems in each interview - Only A-G sequence

3.2.2 Rationale of the interview problems

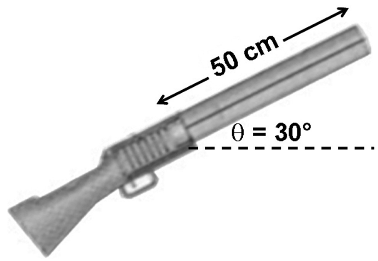
In this chapter, we will analyze students' performance on the algebraic problems in interviews 2 through 7 in both the spring and fall 2009 semesters, because these problems involved integration. The algebraic problem in interview 1 involved calculating kinematics quantities (i.e. velocity, acceleration) by computing the derivative of the position and velocity functions. The algebraic problem in interview 8 involved terms matching between a general function with a specific function for the alternative current in an RLC circuit to find the corresponding quantities. So these two problems will not be discussed in this chapter.

3.2.2.1 Interviews 2 and 3

The algebraic problems in interviews 2 and 3 were simple problems involving the integral concept. In these problems, students had to calculate the work done by non-constant forces by integrating the force functions, i.e. computing $\int F(x)\,dx$. Prior to our interviews, the students had been taught in the lecture that the work done by a non-constant force could be calculated by integrating the force function with respect to the displacement. However, there were no homework or exam problems in which this knowledge was required, so the students did not have a chance to practice the method prior to our interviews. So the algebraic problems in interviews 2 and 3 helped us determine whether or not students could recognize the use of the integral concept in physics problems after they had been taught it but had not practiced on it. These problems helped us answer the research question RQ1.

Figure 3.1 The algebraic problem in interview 2

A 0.1 kg bullet is loaded into a gun (muzzle length 0.5 m) compressing a spring to a maximum of 0.2 m as shown. The gun is then tilted at an angle of 30° and fired.



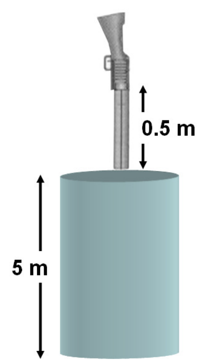
The only information you are given about the gun is that the barrel of the gun is frictionless and that the gun contains a non-linear spring such that when the gun is held horizontally, the net force F (N) exerted on a bullet by the spring as it leaves the fully compressed position varies as a function of the spring compression x (m) as given by:

$$F = 1000x + 3000x^2$$

What is the muzzle velocity of the bullet as it leaves the gun, when the gun is fired at the 30° angle as shown above?

Figure 3.2 The graphical problem in interview 3

A 0.1 kg bullet is loaded into a gun compressing a spring which has spring constant k = 6000 N/m. The gun is tilted vertically downward and the bullet is fired into a drum 5.0 m deep, filled with a liquid.



The barrel of the gun is frictionless. The frictional force F(N) provided by the liquid changes with depth x(m) as per the following function.

$$F = 10x + 0.6x^2$$

The bullet comes to rest at the bottom of the drum
What is the spring compression x?

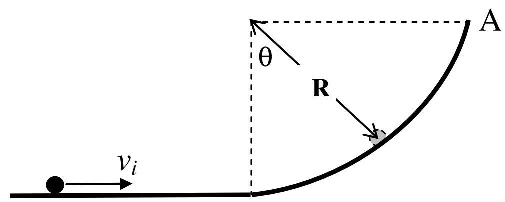
3.2.2.2 Interviews 4

There is a common convention about the integral: we usually use the word "integral" to imply "the integral of the function with respect to its direct variable" unless otherwise indicated. For example, when we say "the integral of $f(x)$," we imply "the integral of $f(x)$ with respect to x," i.e. $\int f(x)dx$. In physics, a physical quantity may be represented as different functions of different variables. So when we talk about "the integral of a force F," for example, we might imply $\int F(x)dx$ if the force F is given as a function of the displacement x, or $\int F(t)dt$ if the force F is given as a function of the time t. This convention has a consequence that if we teach the students that "the work equals the integral of force," without stating which variable the integral should be taken over, then students might claim any integral of force such as $\int F(x)dx$, or $\int F(\theta)d\theta$, or $\int F(t)dt$ as representing the work, while these integrals obviously represented different physical quantities. Students might be able to avoid this error if they think of the total work as an accumulation of infinitesimal work on small segments ds of the trajectory over which the force can be considered constant.

The algebraic problem in interview 4 was created to investigate whether or not students understand what quantity was being accumulated when performing an integral. In this problem, the force was provided as a function of angular displacement instead of linear displacement. This difference made the integral of force with respect to its variable $\int F(\theta)d\theta$ no longer represent the total work done by the force. The total work is the sum of the work on infinitesimal segment of the trajectory, so it must be represented by the integral of force with respect to linear displacement, i.e. $\int F(\theta)ds$, where $ds = Rd\theta$ was an infinitesimal segment of length along the circular track spanning an angle $d\theta$. So to get the correct value of work in this problem, students had to not only integrate the force function, but also multiply the value of that integral by the radius of the track. This procedure was equivalent to calculating the integral $R\int_0^{\pi/2} F(\theta)d\theta$ which equaled to $\int_0^{\pi R/2} F(\theta)ds$. Therefore, this problem required more than just the recognition of the integral concept in the problem. It also required an understanding of what quantity was being accumulated when computing an integral. This problem, therefore, might help us answer the research question RQ2.

Figure 3.3 The algebraic problem in interview 4

A sphere radius r = 1 cm and mass m = 2 kg is rolling at an initial speed v_i of 5 m/s along a track as shown. It hits a curved section (radius R = 1.0 m) and is launched vertically at point A. The rolling friction on the straight section is negligible.



The magnitude of the rolling friction force F_{roll} (N) acting on the sphere varies as angle θ (radians) as per the following function

$$F_{roll}(\theta) = -0.7\theta^2 - 1.2\theta + 4.5$$

What is the launch speed of the sphere as it leaves the curve at point A?

As the students progressed through our interviews, they had become more and more familiar with the use of the integral in physics problems. However, they still had difficulties setting up the integrals from the problem statements, especially when the desired quantities were not the integral of the function with respect to its variable (such as the integral in interview 4). So in the fall 2009 interviews, we continued investigating further the difficulties students had in setting up the integrals by providing them a variety of problems which required more sophisticated understanding of the integral as an accumulation process. The research questions RQ3 will be answered based on the results of this investigation.

Besides investigating students' difficulties, we were also interested in the hints that helped students overcome each of the difficulties. So the research question RQ4 will be answered together with each of the other research questions.

3.2.2.3 Interview 5

There were two algebraic problems in interview 5, which will be referred to as the charged arch problem and the charged rod problem. The charged arch problem involved calculating the electric field due to a charged arch at its center. The charge distribution on the arch was provided as a function of the angular position on the arch. Due to the symmetry of the arch about its vertical axis, only the vertical component of the electric field dE due to each elementary charge on the arch contributed to the total electric field. So the net electric field equaled the integral of the vertical component of dE only, not dE as a whole.

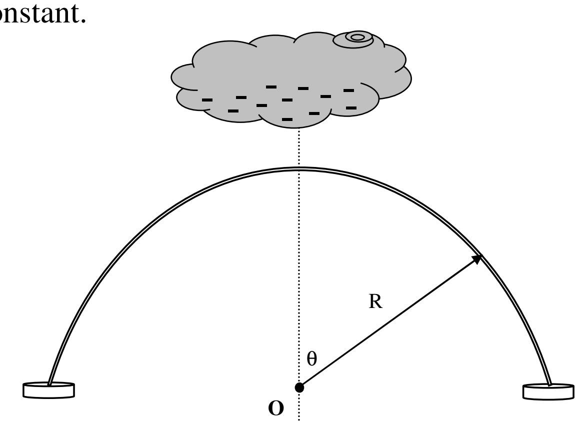
The charged rod problem asked students to calculate the electric field due to a charged rod on which the charge distribution was given as a function of position. Because the electric field dE due to each charge element on the rod points in the same direction, the net electric field is found by integrating dE as a whole.

Figure 3.4 The charged arch problem in interview 5

You are standing at the center of the arch as in problem 1 in a stormy day. There are negatively charged clouds over the arch. The charge distribution λ on the arch now depends on the angle θ as per the function:

$$\lambda(\theta) = \lambda_0 \cos\theta$$

where λ_0 is a positive constant.



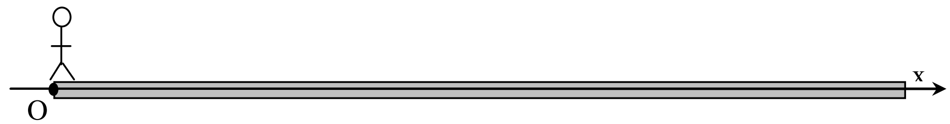
(a) Indicate the charge distribution on the figure below.
(b) Find the magnitude and direction of the electric field at your feet (i.e. at a point O on the ground directly below the top of the arch).

Figure 3.5 The charged rod problem in interview 5

A straight metal rod of length L is lying on the ground but is insulated from the ground. The charge on the rod is distributed with charge density given as per the following function:

$$\lambda(x) = \alpha\, x^2$$

where: α is a positive constant, 'x' is the position on the x-axis relative to the origin O as shown in the figure below.



(a) Indicate the charge distribution on the figure below.
(b) Find the magnitude and direction of the electric field at your feet, located at x = 0

3.2.2.4 Interview 6

There were three algebraic problems in interview 6 which will be referred to as the cylindrical conductor problem, the truncated-cone conductor problem, and the capacitor problem. The conductor problem asked students to find the resistance of a cylindrical conductor whose resistivity was changing along its length. This conductor could be considered as a series combination of several conductors whose length was very small such that the resistivity could be considered constant over that length. The total resistance of the conductor was then the integral of the resistance of each infinitesimal conductor, i.e. $R = \int dR$. The capacitor problem asked students to calculate the capacitance of a circular-plate capacitor. The plates of this capacitor were of different sizes and the separation between the plates was comparable to the diameters of the plates. Due to these conditions, the formula for parallel-plate capacitance that students learned in class, $C = \frac{\varepsilon_0 A}{d}$, was no longer applicable. Instead, this capacitor must be considered as a series combination of capacitors made of fictitious plates, the diameters of which were equaled and were very large compared to their separation. Because the fictitious capacitors were in series, the equivalent capacitance could then be calculated by the integral $\frac{1}{C} = \int \frac{1}{dC}$.

Figure 3.6 The cylindrical conductor problem in interview 6

Find the resistance of a cylindrical conductor of length L, diameter D. The resistivity ρ(x) is changing along the conductor as per the following function:

$$\rho(x) = \alpha x$$

where *x* is the distance from the left end of the conductor.

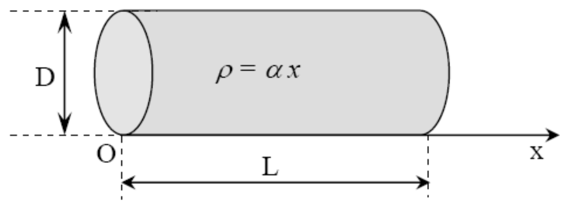


Figure 3.7 The truncated-cone conductor problem in interview 6

A conductor has diameter decreasing from D to d over its length L. The resistivity ρ is constant along the length of this conductor. Find the resistance of this conductor.

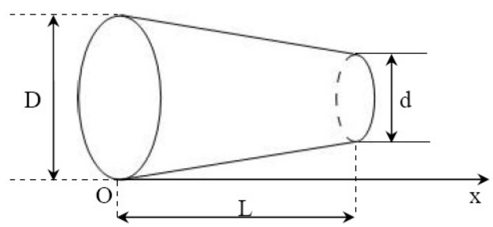
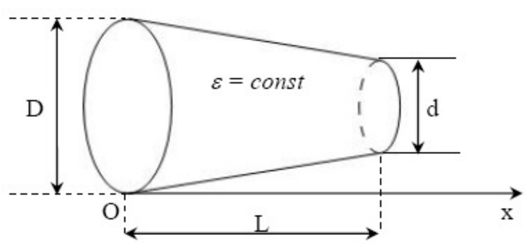


Figure 3.8 The capacitor problem in interview 6

A capacitor is made of two circular conducting plates of diameter D and d. The permittivity ε of the material filled between the plates is constant. Find the capacitance of this capacitor.

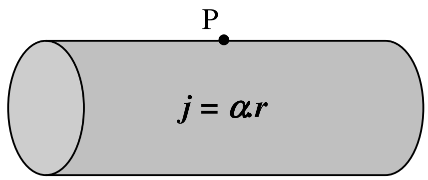


3.2.2.4 Interview 7

The algebraic problem in interview 7 involved finding the total current inside a conducting wire, so it will be referred to as the current problem. The current density in the wire was given as a function of the radial distance from the center of the wire. The total current could be obtained by integrating the currents in the infinitesimally thin rings on the cross-sectional area of the wire. This problem involved integrating in two dimensions.

Figure 3.9 The current problem in interview 7

A cylindrical wire of radius R is carrying a current of density $j = \alpha.r$ (α is a constant, r is the distance from the center of the wire). Find the total current in the wire.



3.2.3 Analysis

Many physics problems involve calculating a physical quantity from other non-constant quantities. Unlike typical problems in calculus courses in which students are given the integrals to compute, physics problems usually do not have pre-determined integrals. The problem statement does not indicate that integrals are needed to solve the problems. Hence, students must be able to recognize the need for an integral and to set up the desired integral from the physics scenario described in the problem statement. For that reason, the first important step in solving a problem is to recognize whether or not a problem requires integration. This step is not trivial for most students because they usually apply the formulas from textbook without noticing the conditions under which those formulas hold. For example, the formula for the work done by a force $W = F \cdot d$ only holds for the case in which the force F is constant over the whole distance d. So if the force is not constant then the work done by the force F must be calculated by the integral $W = \int F \cdot dx$. Similarly, the formula for resistance $R = \rho \frac{L}{A}$ is applicable only for a conductor having constant resistivity ρ and constant cross-sectional area A along its length L, so if ρ or A or both of them are not constant, then the integral must be employed to calculate the resistance. Research by Meredith and Marrongelle, as mentioned in the literature review, reveals the resources that students invoke to cue integration.

The central idea underlying the integral is accumulation, i.e. adding up infinitesimal amounts of a physical quantity to obtain the total amount of that quantity (e.g. resistance) or adding up infinitesimal effects to obtain the total effect (e.g. work, electric field). So to obtain a correct integral, students must have the correct expression for the infinitesimal elements and add up those elements in an appropriate manner (e.g. vectorially, reciprocally). An integral is ready to be computed only after all these steps are done correctly.

In summary, the application of integration in physics problems can be divided into four steps:

- Step 1: recognize the need for an integral
- Step 2: set up the expression for the infinitesimal elements
- Step 3: accumulate the infinitesimal elements
- Step 4: compute the integral

A common theme observed in our interviews was that all students, at some point during the interviews, expressed their understanding of an integral as an accumulation of infinitesimal elements. However, only one or two of them were able to correctly set up and accumulate the infinitesimal quantity without assistance from the interviewer. All other students were not confident in performing the steps and needed guidance through the process.

In this chapter, we will analyze the difficulties students encountered at each of the steps mentioned above. We will start with a general description of the difficulties, then present examples of those difficulties in each of the problems under investigation, and discuss the possible implications from the difficulties.

For the spring 2009 interviews, we only investigated the performance of the students who solved the algebraic problem before the graphical problem (i.e. followed the A-G sequence).

3.3 Results – Spring 2009 – Mechanics

The algebraic problems in the spring 2009 interviews involved calculating the work done by non-constant forces provided as functions of displacement. The formula for the work done by non-constant forces, $W = \int F(x) \cdot dx$, was provided to students during the lecture without its rationale being explicitly addressed. So the students might not understand the accumulation of the infinitesimal work although they could apply the formula to calculate the work.

In the algebraic problem in interviews 2 and 3, the force was provided as a function of linear displacement, i.e. $F(x)$, so students only needed to recall the formula $W = \int F(x) \cdot dx$ to calculate the work. These problems demanded no more than the recall of a previously learned formula. Therefore, in interviews 2 and 3, we only investigate whether or not students could recognize the use of the integral formula in calculating the work (i.e. step 1).

The algebraic problem in interview 4 was more complicated because the force was provided as a function of angular displacement. The integral of force $\int F(\theta)\cdot d\theta$ was not yet the value of work. The correct integral for the total work was $W=\int F(\theta)\cdot ds$ in which $ds=R\cdot d\theta$, so the integral could also be written as $W=R\int F(\theta)\cdot d\theta$. So besides the strategy of thinking about the total work as the sum of infinitesimal works (integrating $\int F(\theta)\cdot ds$), there was an alternative strategy of integrating $\int F(\theta)\cdot d\theta$ then multiplying by the radius R of the track. Because of this alternative strategy, students could also get the correct value for work without understand the structure of the integrand.

3.3.1 Interview 2

There were nine students following the A-G (algebraic – graphical) sequence, which means they were presented with the algebraic problem before being presented with the graphical problem. Conversely, 11 students in this interview followed the G-A (graphical – algebraic) sequence, which means they were presented with the graphical problem before being presented with the algebraic problem. Of the nine students following the A-G sequence:

- Three students spontaneously recognized that work equaled $\int F(x)dx$.
- The six remaining students attempted to calculate the work either by finding the spring constant $k=\frac{F}{x}=1000+3000x$ to plug in the formula for the work done by a spring $W=\frac{1}{2}kx^2$ or by using the formula for the work done by a constant force $W=F.d$ where F was the value of force at maximum compression and d was the distance the bullet traveled. Of these six students, three of them recognized that $W=\int F(x)dx$ after being provided the hint that the spring constant was not a constant, while the other three did not recognize this relationship until the interviewer explicitly told them about it.

3.3.2 Interview 3

There were 11 students following the A-G sequence and nine students following the G-A sequence in this interview. Of the 11 students following the A-G sequence:

- Four students spontaneously recognize that $W = \int F(x)\,dx$.
- Errors that the other seven students made included finding work using $W = F.d$ with F the maximum value of force (2 students), finding work at two ends and averaging (1 student), finding "coefficient of friction" from the algebraic expression of $F(x)$ (1 student). The other three students said that they knew that work was either the derivative or the integral of force but did not know specifically which one. Of the seven students who had errors, five recognized $W = \int F(x)\,dx$ after being hinted by the interviewer, while two did not recognize it until they were explicitly told so by the interviewer. The hint provided in this interview was to guide students to think of the total work as the sum of works on small segments of the path.

3.3.3 Interview 4

There were 11 students following the A-G sequence and nine students following the G-A sequence in this interview. Of the 11 students following the A-G sequence:

- All 11 students recognized that they had to integrate the force function.
- Only one of them spontaneously recognized that he must have $\int F(\theta)\,ds$ instead of $\int F(\theta)\,d\theta$.
- Five students calculated the integral of force $\int F(\theta)\,d\theta$ and multiplied by the total distance.
- Five other students just calculated $\int F(\theta)\,d\theta$ and claimed that it was the value of work.

All of the 10 students who had errors were able to recognize that they had to either take $\int F(\theta)\,ds$ or convert the unit after taking $\int F(\theta)\,d\theta$ to get the correct value of work after several hints were given by the interviewer. For students who did not know what to do with the force function or used it in an incorrect way, then the hint was to guide them to think of the total work as the sum of infinitesimal work $dW = F \cdot ds$. For students who integrated $\int F(\theta) \cdot d\theta$, the interviewer let them continue along this path until they got the value of the integral. The students were then asked about the unit of their integral, and when they recognized that they did not have the correct unit of work, the interviewer assisted them in doing the unit conversion, during which they recognized the need for the radius factor.

3.3.4 Conclusion from the spring 2009 interviews

The algebraic problems in the spring 2009 interviews involved calculating the work done by non-constant forces provided as a function of displacement. We found that not many students were able to recognize that the work equaled the integral of force, although they had learned about it in the course. Many of them attempted to use pre-derived formula for the work done by a force $W = F \cdot d$ or spring force $W = \frac{1}{2}kx^2$ even when these formulas did not apply. Even when the students spontaneously recognized that work equaled the integral of force, there was evidence suggesting that they might just recall it from what they learned in the course without an understanding of the underpinning of the method. First, only one student in interview 4 spontaneously recognized that the integral was not the value of the work. The rest of the students simply calculated the integral and claimed that it was the value of work, which might imply their lack of understanding of how the work was accumulated when they performed the integral. Second, there were some students who realized that they had to either differentiate or integrate the function but did not know which one to do. This fact indicated that most students simply remembered the strategy without understanding the underlying process of integration.

We answer the research questions RQ1, RQ2, and RQ4 as follows.

RQ1: Did students recognize the use of the integral in physics problems?

Most of the students were not able to recognize the use of the integral in calculating the work done by non-constant forces. Instead, they attempted to use pre-derived formulas to calculate the work. Students' inability to recognize the use of the integral might be attributed to their unfamiliarity with the task (since students did not have any problems involving integral prior to our interviews) and their strong inclination to using the pre-derived formulas rather than attempting an unfamiliar strategy or inventing a new strategy.

RQ2: Did students understand what quantity was being accumulated when calculating an integral?

The fact that some students knew that they had to calculate the derivative or the integral of force but did not know which one suggested that these students did not understand the physical meaning of the operators. Therefore, students' application of the integral in finding work might simply be the recall of the previously learned knowledge (i.e. the work equaled the integral) rather than an understanding of how the work was being accumulated.

The fact that most of the students claimed the integral $\int F(\theta)\, d\theta$ in interview 4 as the value of force indicated that these students did not understand what quantity was being accumulated when they performed the integral.

RQ4: What verbal hints may help students overcome those difficulties?

For students who attempted to use pre-derived formulas they learned from the course to calculate the work, the hints were to help them recognize that those formulas were not applicable to the problems at hand. For example, when a student attempted to find the spring constant using $k = \frac{F}{x}$ to plug in the formula $W = \frac{1}{2}kx^2$, the hint was to ask them whether the spring constant was actually a constant, which helped them recognize that the concept of "spring constant" did not apply for non-linear spring and hence the formula $W = \frac{1}{2}kx^2$ did not apply either. The hints that guided students to think of the non-constant nature of the force triggered students' thinking of integration. The hints on the accumulation of the infinitesimal work to get the total work also helped some students to set up the correct integral for the work in interview 4, although the hints on units seemed to be easier to understand for the students.

3.4 Results – Fall 2009 – Electricity and Magnetism

The use of the integral concept in electricity and magnetism (E&M) is more intensive and complicated than in mechanics. There were no pre-derived formulas for calculating E&M quantities using the integral concept that students had learned from the lecture as there was for work in mechanics. So to successfully set up the integral representing a quantity in E&M, students must understand how the infinitesimal quantity is calculated and accumulated.

In each of the following sub-sections, we will discuss the difficulties students encountered at each of the steps of applying the integral concept to physics problems described in the previous section:

- Step 1: recognize the need for an integral
- Step 2: set up the expression for the infinitesimal elements
- Step 3: accumulate the infinitesimal elements
- Step 4: compute the integral

We will begin with a general description of the difficulties and then present examples of those difficulties in each of the problems under investigation.

3.4.1 Students' recognition of integration

Most of the students in our interviews did not have significant difficulty recognizing the need for the integrals in the interview problems. We observed that the non-constant physical quantity given in the problem statement was the major cue for integration.

The charged arch problem (Figure 3.4) and *charged rod problem* (Figure 3.5) were very similar to some of the homework and exam problems in the course, so all students knew that they had to use the integral to calculate the electric field.

On *the cylindrical conductor problem* (Figure 3.6), 12 out of 15 students stated, with different levels of confidence, that an integral was needed because the resistivity was changing along the conductor. The reasoning provided by student S6 "*since ρ isn't constant we're going to have to do an integral*" was typical for students who were confident with their reasoning. On the other hand, the question posed by student S13, after setting up the expression $\frac{\alpha x L}{A}$, "*Do I have to put an integral somewhere?*" indicated her uncertainty about the need for an integral. The remaining three students also arrived at the expression $\frac{\alpha x L}{A}$ and claimed that it was the final answer. When the interviewer hinted that the final answer should not contain x, these students were able to recognize that they needed an integral. The following excerpt was typical among this group of students.

Interviewer: Is this [$\frac{\alpha x L}{A}$] your final answer?

S3: Uh … yes.

Interviewer: But that answer contains x which is changing.

S3: Okay … so … should I use integration?

The truncated-cone conductor problem (Figure 3.7) followed the cylindrical conductor problem in interview 6. Thirteen out of 15 students were able to recognize that they could use the integral set up in the cylindrical conductor problem except that the area was then a variable. The other two students wrote an integral with dA – the infinitesimal cross-sectional area – as the infinitesimal term, i.e. $\int \frac{\rho(x)L}{dA}$.

The capacitor problem (Figure 3.8) was the last problem in interview 6. Only 12 out of 15 students got to this problem within the one-hour time limit of the interview. All of them were able to recognize the need for an integral to solve the problem.

The current problem (Figure 3.9) was asked in interview 7. Thirteen out of 15 students stated that they needed to have an integral to calculate the total current. The other two students attempted to find the total current by multiplying the current density at the surface of the wire by the total cross-sectional area of the wire.

In conclusion, we found that most of the students could easily recognize the need for an integral in the problem. The presence of the non-constant quantities was the hint for students to think of using integration. This finding agrees with the finding of Meredith and Marrongelle (2008) that the dependence cue was most commonly used by students to cue integration in physics problems.

3.4.2 Set up the expression for the infinitesimal quantities

In order to calculate an integral, one must know the variable of integration. One way to do that is to look at the infinitesimal term (e.g. dx, dr, $d\theta$, …) in the integral. This term also carries a physical meaning that must be understood while setting up the integral. For example, if $F(x)$ is a function of force with respect to position x, then $\int F(x)dx$ means integrating the product of the force $F(x)$ at position x and the corresponding infinitesimal distance dx, in the direction of the force to obtain the total work done over the whole distance. On the other hand, $\int F(t)dt$ means integrating the product of the force $F(t)$ at time t and the corresponding infinitesimal time interval dt to obtain the total impulse due to the force over the total time interval. In these examples, dx and dt not only indicate the variable of integration but also have their own physical meanings: infinitesimal distance and infinitesimal time interval. So it is mathematically incomplete and physically meaningless to write the integral as $\int F(x)$. However, it was observed that many students in our interviews either set up the integral without the infinitesimal term or simply appended it to the integrand or to whatever quantity that was changing. These actions essentially changed the physical meaning of the integrand.

The charged arch problem (Figure 3.4): Starting with the formula for the electric field due to a point charge $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$, all 15 students were able to write the electric field due to a charge element dq as $dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$.

The charged rod problem (Figure 3.5): This problem followed the charged arch problem in the same interview. After doing the charged arch problem, all students knew that they had to integrate $dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$.

The cylindrical conductor problem (Figure 3.6): To solve this problem, one must set up the expression $dR = \rho(x)\frac{dx}{A}$ for the infinitesimal resistance of a thin slice of the conductor, then integrate to find the total resistance $R = \int_0^L \rho(x)\frac{dx}{A}$, where A is the constant cross-sectional area of the conductor. Eight out of 15 students started with the formula of resistance $R = \rho\frac{L}{A}$ and then set up the integral $R = \frac{L}{A}\int \rho(x)$ or $R = \frac{L}{A}\int \rho(x)\,dx$. The first integral was mathematically incomplete and the second integral did not represent any physical quantity. Among the remaining seven students, one student recognized that she needed an infinitesimal length dL in place of L in the formula, three students recognized this after being reminded of the meaning of L in the integral, and the other three students were able to set up the integral only after detailed guidance from the interviewer.

The truncated-cone conductor problem (Figure 3.7): Twelve out of 15 students stated that they could use the integral set up in the cylindrical conductor problem but with area being a variable. These students could also recognize that since there were two variables in that integral: x and A, they had to write one variable in terms of the other in order to integrate. All students needed a lot of guidance from the interviewer in writing area A in terms of x. One student set up the correct integral but stated that the limits of integral were from d to D because the diameter was changing. Being hinted that dx indicated integration with respect to x, hence the limits should be the range of x, this student recognized that the limits were from 0 to L. Therefore, we interpret this student's wrong choice of limits as evidence that she did not understand that dx indicated the integration variable x. Two other students set up the integral for resistance as $R = \int_{\pi\left(\frac{d}{2}\right)^2}^{\pi\left(\frac{D}{2}\right)^2} \frac{\rho L}{dA}$. These students stated that because area A was changing, they used the infinitesimal area dA. Obviously, the term $\frac{\rho L}{dA}$ did not represent the infinitesimal resistance of a thin slice of the conductor.

The capacitor problem (Figure 3.8): To solve this problem, students must think of a capacitor with large separation between the plates as a combination of several capacitors made of fictitious plates separated by an infinitesimal distance dx. This strategy was novel to many students, so they needed hints to recognize the idea. After the hints were provided, 10 out of 12 students were able to set up the correct expression for the capacitance of a capacitor with infinitesimal separation dx between the plates: $dC = \varepsilon \frac{A(x)}{dx}$. The other two students used the differential area dA and got $dC = \varepsilon \frac{dA}{L}$. This error was similar to the error observed in the truncated-cone resistor problem, where students had $\frac{\rho L}{dA}$ as the infinitesimal resistance. This type of error suggested that these students seemed to simply prefix "d" to whatever quantity that was changing (i.e. area A in these cases) without understanding the meaning of the infinitesimal term in the integral.

The current problem (Figure 3.9): The correct expression for the infinitesimal current in the wire is $j(r)dA$, where $j(r)$ is the current density at a distance r from the center of the wire and dA is the area of an infinitesimally thin ring on the cross-section of the wire. Thirteen out of 15 students made mistakes similar to those observed in the cylindrical conductor problems: they set up $I = A\int j(r)$ or $I = A\int j(r)dr$, where A was the total cross-sectional area of the wire. When the interviewer reminded the students about the formula $I = \int j(r)dA$, all of the students agreed that they had seen it before but then failed to tell what dA meant in that formula.

In conclusion, we found that students' failure in setting up the expression for the infinitesimal quantity was due to their lack of understanding of the physical meaning carried by the infinitesimal term (e.g. dx, dr, $d\theta$ …). This lack of understanding caused students to ignore the infinitesimal term or to simply append it to the integrand, or even to prefix d to whatever quantity that was changing when setting up the expression for the infinitesimal quantity. All of these actions essentially changed the physical meaning of the expression being set up.

3.4.3 Accumulating the infinitesimal quantities

It was observed in our interviews that after having a correct expression for the infinitesimal quantity, almost all of the students started integrating that expression without attention to how these quantities should be added up.

The charged arch problem (Figure 3.4): The electric fields dE due to the infinitesimal elements of charge on the arch must be added vectorially. Eight out of 15 students in our interview did not notice the vector nature of dE and integrated dE as a whole, while the other seven students used symmetry to argue that only the y-component of the electric field due to each charge element contributed to the total field and integrated only the y-component of dE.

The charged rod problem (Figure 3.5): The electric fields dE due to all infinitesimal elements of charge dq on the rod were pointing in the same direction so the total field could be obtained by simply integrating dE. So even though all of the students could do this step, we could not conclude whether they understood that they were adding vectors having the same direction or just integrated the infinitesimal quantity.

The cylindrical conductor and *the truncated-cone conductor problems* (Figure 3.6 and Figure 3.7): The slices that made up the conductor were connected in series, so the total resistance could be obtained by adding up the resistance of these slices. When the thickness of each slice became infinitesimally small, this was accomplished by integrating dR. Similarly, in *the current problem* (Figure 3.9), because the currents in all thin rings that made up the cross section of the wire were in the same direction, the total current could be obtained by integrating the infinitesimal current dI in each ring. In these cases, the total quantities were obtained by simply integrating the infinitesimal quantities, i.e. $R = \int dR$ and $I = \int dI$, so we could not conclude whether or not students really understood how the infinitesimal quantities should be accumulated.

The capacitor problem (Figure 3.8): The capacitor in this problem could be viewed as a series of capacitors whose plates were separated by a small distance. The equivalent capacitance could be found by adding the capacitance of each individual capacitor reciprocally, i.e. $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots$ which became $\frac{1}{C_{eq}} = \int \frac{1}{dC}$ when the separation between the plates became infinitesimally small. This problem demanded more than just integrating the infinitesimal quantities to obtain the total quantity. It also required an understanding of integration in association with the physical situation of the problem.

Out of 12 students who attempted the capacitor problem, only two students recognized that they had to integrate $\frac{1}{dC}$. The other 10 students integrated dC and got the integral $C = \int dC = \int_0^L \varepsilon \frac{A(x)}{dx}$. These students immediately recognized that this integral had dx in the denominator, so they attempted to bring dx to the numerator although they could not give a reason why they could do that. The interviewer had to give hints to cue students' attention to the arrangement of the capacitors. The following excerpt is typical in this situation.

Aaron: … since L is going to turn into dx I think … but to make that … it should be dx in the denominator … [wrote $\int_0^L \varepsilon \frac{\frac{\pi d^2}{4}}{dx}$ and then flipped the integrand]

Interviewer: Why did you flip it?

Aaron: Well, so that dx is in the numerator.

Interviewer: You must have a reason for flipping the integrand.

Aaron: Oh, okay …

Interviewer: What does your integrand mean?

Aaron: Like if you slice it up it's just one of the slices.

Interviewer: Okay, but when you add up capacitance, you must know how the capacitors are connected, that is, in parallel or in series.

Aaron: Um … it doesn't say.

Interviewer: Look at how the plates are arranged.

Aaron: Um …

Interviewer: You should draw some of the fictitious plates to see how they are arranged.

Aaron: [draws the plates] Okay … so … they are in series, aren't they?

Interviewer: Yes, and what is the equation for capacitors in series?

Aaron: It's the one over thing.

Interviewer: So how should you integrate in this problem?

Aaron: Well … because integral means sum … and I have … so the integral is … [writes $\int \frac{1}{dC}$]

In this excerpt, Aaron indicated an understanding of the meaning of the integrand ("*if you slice it up it's just one of the slices*"), the structure of the integrand (i.e. dx must be in the numerator), and the formula for capacitors in series ("*it's the one over thing*"). However, he was not able to recognize that the capacitors were in series until he drew the fictitious plates between the two plates of the capacitor. Similar situations also occurred with other students who integrated dC. This evidence suggested that students' lack of visualization of the physical scenario might account for their disregard of how the quantity should be accumulated.

3.4.4 Computing the integral

The last step in applying integration to physics problems is to compute the integral set up in the previous three steps. This was expected to be an easy task for students because they had practiced computing integrals in their calculus courses. However, students still had some difficulties with computing the integrals in our interview problems.

The charged arch problem (Figure 3.4): Upon having the integral for the electric field due to the arch $E = \int \frac{1}{4\pi\varepsilon_0} \frac{dq}{r^2} \cos\theta$, 13 out of 15 students were unable to recall the relation $dq = \lambda ds$ between the charge element dq and the length ds of that element along the arch. Eleven out of 15 students could not relate infinitesimal length of the arc to the infinitesimal angle it subtended at the center: $ds = rd\theta$. After the variable conversion, the resulting simplified integral was $\int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta$. All 15 students needed to be given the equation $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ and two of them needed assistance in computing the integral explicitly.

The charged rod problem (Figure 3.5): Students' difficulties with computing the integral in this problem were due to students' inability to interpret the physical meaning of symbols. Twelve out of 15 students interpreted r in Coulomb's law as "radius," so they were unable to decide whether r was a constant or a variable in the integral. The charged rod problem came right after the charged arch problem, so all students were then able to write $dq = \lambda ds$, but then 11 of them were not able to recognize that $ds = dx$ in this problem.

The cylindrical conductor problem (Figure 3.6): The integral in this problem was very simple so all students were able to compute it without assistance from the interviewer.

The truncated-cone conductor and the capacitor problems (Figure 3.7 and Figure 3.8) The most difficult part of computing this integral was to figure out the expression for the cross-sectional area as a function of position. However, because it was not the purpose of the interview to test students' geometric skills, the expression for $A(x)$ was provided to the students if they failed to get it after a few attempts. The resulting simplified integral was $\int_0^L \frac{dx}{\left(D + \frac{d-D}{L}x\right)^2}$ where D, d, L were constants. Only two students succeeded in computing this integral using substitution. Others needed to be given the result of the integral. In the truncated-cone conductor problem, one student set the limits of the integral as d and D (i.e. the diameters of the conductor at two ends) based on the fact that the diameter was changing. The same error was made by five students when solving the capacitor problem, including those who had the correct limits for the integral in the truncated-cone conductor problem.

The current problem (Figure 3.9): The most difficult part of computing the integral in this problem was to write the differential cross-sectional area dA in terms of the distance r from the center of the wire. Asking students to take the derivative of the cross-sectional area $A = \pi r^2$ helped students derive the expression $dA = 2\pi r dr$. The resulting integral was very simple, so all of the students were able to compute it.

In summary, we found that students encountered a number of difficulties in computing the integrals in physics problems. Some of these difficulties could be attributed primarily to students' misunderstanding of the physical meaning of symbols in the integrals. Other difficulties arose when students could not recall basic mathematical equations. A few students still had difficulties determining the limits of the integrals. Many students were unable to compute mathematical integrals.

3.4.5 Conclusion from the fall 2009 study

In this study, we took a close look at students solving problems involving integration in the context of electricity. We found that students' failure in applying integration to our interview problems occurred when students set up the expressions for the infinitesimal quantities and accumulated those quantities using integral. These difficulties might be attributed primarily to students' inability to interpret the meaning of the infinitesimal term dx in the integral, and to students' disregard of how the quantities must be added up. A few students still had difficulties recognizing when an integral was needed in a problem. Students also had difficulties in computing the integrals they had set up, mostly because they were unable to interpret the physical meaning of the symbols and to invoke basic mathematical equations.

We answer the research question RQ3: What are the common difficulties that students encounter when solving problems in electricity involving integration? Students generally did not have significant difficulty recognizing the need for integration in a problem. However, students did have significant difficulties setting up and computing the desired integral. These difficulties included setting up an incorrect expression for the infinitesimal quantity and/or accumulating the infinitesimal quantities in an inappropriate manner. Determining the limits of the integrals, relating variables in an integral, and computing the integrals algebraically were also the difficulties faced by some of the students.

These findings align with those from other research on students' difficulties with integration. We found that the non-constant quantity given, either mathematically (e.g. resistivity as a function of position, charge distribution as a function of angle) or pictorially (e.g. figure of a conductor with changing diameter), in the problem statement was the cue for most students to think of integration in a problem. This finding supports the conclusion of Meredith and Marrongelle (2008) that the most common resource that students use to cue integration is the dependence cue. However, the dependence cue, as pointed out by Meredith and Marrongelle, is only helpful when the non-constant quantity is a density or a rate of change. This finding also aligns with the fact that many students in our study failed to set up the correct integral in problems involving non-constant quantities which were not rates of change (e.g. resistivity, diameter).

Although most of the students indicated an understanding of integration as an accumulating process, they were not confident in carrying out the process and needed detailed guidance from the interviewer. Some of the students had difficulties determining the limits of integral. These observations are similar to those described by Cui *et al.* (2006).

Our study extends the literature on students' use of integration in physics problem solving. We found that the major difficulties students encountered when attempting to set up an integral in a physics problem were due to students' inability to understand the infinitesimal term in the integral and failure to understand the notion of accumulation of an infinitesimal quantity.

Meredith and Marrongelle (2008) suggested that the parts-of-a-whole symbolic form was a more powerful and flexible resource to cue integration and proposed instructional strategies to promote students' use of this recourse as a cue for integration in physics problems. Our study pointed out that setting up a correct integral in a physics problem requires more than recognizing the need for an integral. It also requires setting up the correct expression for the infinitesimal quantity that each "part" represents and accumulating that quantity in a correct manner. There were several students in our interviews who mentioned the sum of infinitesimally small elements (although they did not use that terminology) at some point while solving the problems, indicating that they had a rough idea of the parts-of-a-whole resource, but then set up the incorrect expression for the "part" or did not pay attention to how the "parts" should be added up. So we expand upon the conclusion of Meredith and Marrongelle that although the parts-of-a-whole symbolic form is the most powerful and flexible way to think of integration, it does not guarantee the correctness of the integral that is set up.

3.5 Limitations and future work

The research methodology used in both the spring 2009 and fall 2009 studies was individual interview. This method allowed us to gain detailed insight into students' performance on the problems and also enabled us to interview the same students several times on different topics during the two semesters. On the other hand, the individual interview method limited the number of student participants in the study. There were only 20 students in the spring 2009 study and 15 students in the fall 2009 study compared to more than 200 students enrolled in each of the courses from which the interviewees were recruited. Due to this fact, the major limitation of this study is the generalizability of its findings.

Based on our interview findings, we plan to develop tutorial materials to address students' difficulties with integration in physics problems and implement them with all of the students in the course (usually around 200+ students) in future semesters when the courses are offered to test the effects of those materials in helping students learn to solve physics problems involving integration.

Chapter 4 - Students' application of the area under the curve concept in physics problems

4.1 Introduction

In this chapter, we investigate how students solve physics problems in which part of the information is given as a graph. Specifically, we look at how student use the area under the curve concept to calculate a physical quantity. We examine the following research questions:

- RQ1: To what extent did students recognize the use of the area under the curve in physics problems?
- RQ2: To what extent did students understand what quantity was being accumulated when calculating the area under a curve?
- RQ3: To what extent did students understand the relationship between a definite integral and the area under a curve?

We will analyze students' performance on the graphical problems in interviews 2 through 7 in the spring and fall 2009 semesters. In interviews 2 through 4, the graphical problems involved calculating the work done by non-constant forces from the graphs force functions. In interviews 5 through 7, the graphical problems involved computing pre-determined integrals using the area under the curve concept. The graphical problem in interview 1 involved calculating kinematics quantities (i.e. velocity, acceleration) by computing the slope of the curve. The graphical problem in interview 8 involved reading out information from a graph. So these two problems will not be discussed in this study.

4.2 Rationale of the interview problems

The interview problems were designed to help us answer the research questions mentioned above. In the spring 2009 interviews, our problems aimed at exploring whether or not students could recognize the use of the area under the curve concept in physics problems, and whether or not students understood what quantity the area under the curve represented. These problems helped us answer the first two research questions:

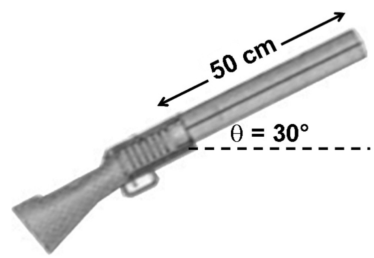
- RQ1: To what extent did students recognize the use of area under the curve in physics problems?

- RQ2: To what extent did students understand what quantity was being accumulated when calculating the area under a curve?

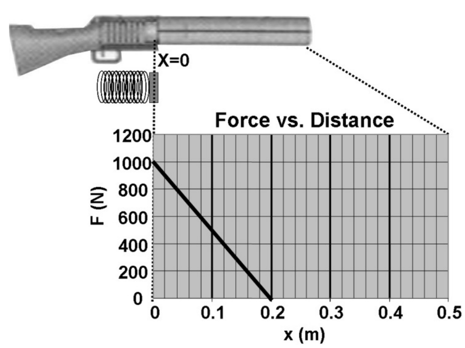
In the graphical problems in interviews 2 and 3 (Figure 4.1 and Figure 4.2) students had to calculate the work done by non-constant forces by evaluating the area under the curve of force versus linear displacement. Prior to our interviews, the students had been taught in the lecture that the work done by a force could be calculated using area under the curve of force versus displacement. However, there were no homework or exam problems in which this knowledge was required, so the students did not have a chance to practice the method prior to the interviews. So the graphical problems in interviews 2 and 3 might help us determine whether or not students could recognize the use of the area under the curve concept in physics problems after they had been taught it but had not practiced on it.

Figure 4.1 The graphical problem in interview 2

A 0.1 kg bullet is loaded into a gun (muzzle length 0.5 m) compressing a spring as shown. The gun is then tilted at an angle of 30° and fired.



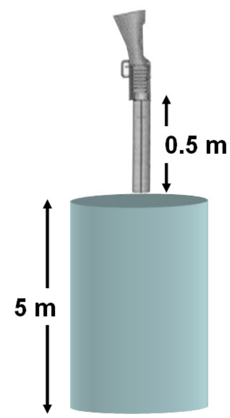
The only information you are given about the gun is that the barrel of the gun is frictionless and when the gun is held horizontally, the net force F (N) exerted on a bullet by the spring as it leaves the fully compressed position varies as a function of its position x (m) in the barrel as shown in the graph below.



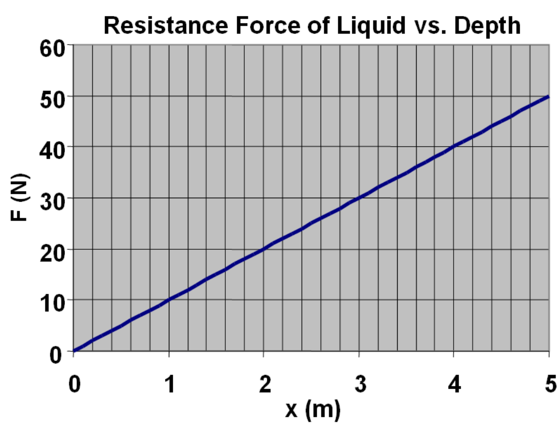
What is the muzzle velocity of the bullet as it leaves the gun, when the gun is fired at the 30° angle as shown above

Figure 4.2 The graphical problem in interview 3

A 0.1 kg bullet is loaded into a gun compressing a spring which has spring constant k = 6000 N/m. The gun is tilted vertically downward and the bullet is fired into a drum 5.0 m deep, filled with a liquid.



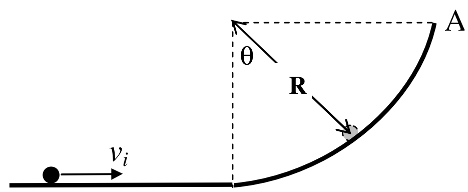
The barrel of the gun is frictionless. The resistance force provided by the liquid changes with depth as shown in the graph below. The bullet comes to rest at the bottom of the drum. What is the spring compression *x* ?



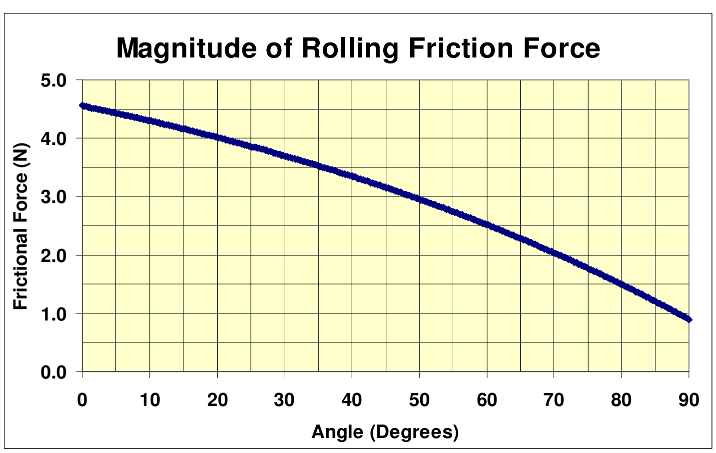
The graphical problem in interview 4 (Figure 4.3) was designed to answer the research question RQ2. This problem also involved finding the work done by a frictional force using the area under the curve concept. However, the graph provided in this problem was the graph of force versus angular displacement instead of linear displacement, so finding the area under the curve meant accumulating the product of force and angle, which did not yield the total work. Students had to convert angular displacement into linear displacement along the circular track by multiplying the angular displacement by the radius of the track. So to get the correct value of work in this problem, students had to not only calculate the area under the curve, but also multiply that area by the radius of the track. This procedure was equivalent to calculating the integral $R\int_{0}^{\pi/2} F(\theta)\,d\theta$ which equaled to $\int_{0}^{\pi R/2} F(\theta)\,ds$, where $ds = Rd\theta$ was an infinitesimal segment of length along the circular track spanning an angle $d\theta$. Therefore, this problem required more than just the recognition of the area under the curve concept. It required an understanding of what quantity was being accumulated when computing the area under the curve. Without such an understanding, students might claim the area under the curve itself as the value of the work. This problem, therefore, could help us determine whether students understood what physical quantity the area under the curve represented or just applied the knowledge of "work equaled area under the curve of force" without understanding its underpinnings.

Figure 4.3 The graphical problem in interview 4

A sphere radius r = 1 cm and mass m = 2 kg is rolling at an initial speed v_i of 5 m/s along a track as shown. It hits a curved section (radius R = 1.0 m) and is launched vertically at point A. The rolling friction on the straight section is negligible.



The magnitude of the rolling friction force acting on the sphere varies as angle θ as per the graph shown below. What is the launch speed of the sphere as it leaves the curve at point A?



As students proceeded through our interviews in the spring 2009 semester, they had become familiar with the use of the area under the curve concept in physics problems. Another issue arose: Most of the students could state that "the integral equaled the area under the curve," but did they know which curve? In other words, when students had an integral and more than one graphs, did they know which area under the curve was equal to the integral? Obviously, in order to choose from several graphs the one corresponding to a pre-determined integral, students had to understand the relationship between integral and area under the curve. So our problems in the fall 2009 interviews were designed to help us answer the third research question:

RQ3: To what extent did students understand the relationship between a definite integral and an area under a curve?

In each of these problems, students had to calculate a physical quantity (e.g. electric field, resistance, electric current) by evaluating a definite integral. Explicit expression of the integrand or part of the integrand was not given. Instead, students were provided with several graphs of quantities related to the integrand. Students had to choose the graph on which the area under the curve equaled the integral at hand. These problems could help us determine whether students understood how a definite integral was related to an area under a curve.

In the next sections, we will present our findings from the interviews and discuss how these findings help us answer our research questions. We will use pseudonyms S1 to S15 to identify the students.

4.3 Results – Spring 2009 – Mechanics

4.3.1 Students' recognition and understanding of the area under the curve concept

The graphical problems in interviews 2, 3, and 4 involved calculating the work done by non-constant forces from the graphs of force versus linear or angular position. We found that in interviews 2 and 3, most of the students attempted to calculate the work by using pre-determined formulae for the work done by constant forces. Upon being asked to think of another strategy to find work, only a few students were able to recognize that they could instead calculate the area under the curve of force. Other students only recognized the use of the area under the curve after hints or detailed guidance were provided by the interviewer. In interview 4, students had become familiar with the task, so most of them spontaneously stated that work equaled the area under the curve of force. However, as discussed above, students had to multiply the area under the curve by the radius of the circular track to obtain the correct value of the work done by the rolling friction force. Only one student could recognize the need for the radius factor without assistance from the interviewer. Many other students did not understand why they needed a radius factor even after hints or detailed guidance were provided by the interviewer.

We classified students' performance into three levels:

- getting the correct answer spontaneously, i.e. without hints from the interviewer.
- getting the correct answer after a few hints given by the interviewer. The hints given were to ask students to think about the structure of the equation for work or its unit: Work is the product of force and displacement, and the unit of work is the product of units of force and displacement. Then students were asked to think about how such a product could be obtained from the graph (i.e. multiplying the quantities on the vertical and horizontal axes, which essentially yielded the area).
- getting the correct answer after detailed, step-by-step guidance from the interviewer.

We will now discuss students' performance on each of the problems.

4.3.1.1 Interview 2

The graphical problem in this interview involved finding the work done by a spring. There were two possible strategies for calculating the work done by the spring force in this problem:

- finding the area under the curve of force
- finding the spring constant k. Because of the linear dependence of spring force on displacement ($F = -kx$) in this problem, the spring constant k equaled the magnitude of the slope of the line. Then the work done by the spring force could be found from the equation $W = \frac{1}{2}kx^2$, where x was the maximum spring compression.

Only one out of 11 students following the G-A sequence spontaneously stated that work equaled the area under the curve of force versus distance, and used the first strategy to calculate work. The other 10 students followed the second strategy and also obtained the correct value of work. When these students were asked to think of another strategy to find the work done by the spring force, 6 students could recognize that work equaled area under the curve of force after hints. The other 4 students stated that the area might have a physical meaning but were not able to recall what the meaning was until being told explicitly by the interviewer.

4.3.1.2 Interview 3

The graphical problem in this interview involved finding the work done by the resistance force of a liquid. This work might be found by finding the area under the curve of force. Only 3 out of 9 students who followed the G-A sequence spontaneously stated that work equaled area under the line. Three other students attempted to use the equation for the work done by the frictional force on a horizontal floor $W = F \cdot d = \mu mgd$ in which the coefficient of friction μ was the slope of the curve. Another student stated that the slope of the curve was the value of work. The remaining 2 students attempted to use the equation $W = F \cdot d$ where F was the value of force at the maximum point on the graph. Of the 6 students who did not spontaneously calculate area under the curve, 3 recognized that work could be calculated using the area under the curve after hints, while the other 3 were not able to recognize it until being told explicitly by the interviewer.

4.3.1.3 Interview 4

This problem involved finding the work done by the rolling friction force on a circular track. This could be done by finding the area under the curve and multiplying this area by the radius of the track. Only one out of 9 students following the G-A sequence spontaneously set up the correct calculation and obtained the correct value for the work. Five other students spontaneously stated that the area under the curve was the value of work. Of these 5 students, upon being told that the area itself was not the value of work, only 2 students recognized the need for the radius factor while the other 3 students did not know what was missing and needed detailed guidance from the interviewer. The remaining three students needed detailed guidance on both recognizing the use of the area under the curve and the need for the radius factor.

4.3.2 Conclusion from the spring 2009 study

Table 4.1 summarizes the number of students (out of the total) who obtained the correct value of work using area under the curve without hints, with hints, and with detailed guidance.

Table 4.1 Students' performance in the spring 2009 interviews

Interview	**Correct without hints**	**Correct after hints**	**Correct after detailed guidance**
2	1/11 S16	6/11 S1, S2, S4, S8, S19, S14	4/11 S3, S7, S11, S12
3	3/9 S16, S10, S15	3/9 S2, S6, S12	3/9 S3, S11, S13
4	1/9 S9	2/9 S6, S10	6/9 S5, S17, S18, S13, S15, S20

From Table 4.1, we see that only a few students (S9, S10, S15, S16) could spontaneously recognize the use of the area under the curve in calculating work when the graph of force versus displacement was provided. Student S16 followed the A-G sequence in interview 4 so he was not included in the analysis of the graphical problem for this interview. In this problem, both of the students S10 and S15 spontaneously stated that the work done by the rolling friction force was the area under the curve of force versus angle. One of them (S10) could recognize the need for the radius factor after being told that the area itself was not the value of work. The other student (S15) only obtained the correct value of work after detailed guidance from the interviewer. Student S9 was the only one who could calculate the correct value of work in the graphical of interview 4 without any assistance. However, he followed the A-G sequence in interviews 2 and 3, so he was not included in the analysis of the problems in those interviews..

We answer our first two research questions as follows.

- RQ1: To what extent did students recognize the use of area under the curve in physics problems?

The majority of students in our interviews did not spontaneously recognize the use of area under the curve in calculating work from the graph of force. There were two possible explanations: (i) students were not familiar with the method; and (ii) students held strong preference on algebraic method. The fact that more students were able to recognize that work equaled the area under the curve as they progressed through the interviews suggested that students gained familiarity with the concept. Some students, while talking to the interviewer after the interviews, stated that they had not seen any problem using the area under the curve in their physics homework or exam. On the other hand, students also expressed an inclination to an algebraic approach even when a graph was provided. They attempted to use pre-derived formulae for work and just used the graph to collect data on the values of spring constant or coefficient of friction to plug in those formulae. Some students explicitly told the interviewer that they hated problems with graphs and preferred working with equations. These facts supported the second explanation.

- RQ2: To what extent did students understand what quantity was being accumulated when calculating the area under a curve?

In the graphical problems in interviews 2 and 3, the area under the curve itself was the value of work. So when a student recognized that work equaled the area under the curve, we did not know whether he understood how work was accumulated when calculating the area or he just applied what he was taught in the lecture. There were four students in interview 2 stated that the area had some meaning but were not able to tell what the meaning was, and three students in interview 3 stated that the slope of the line was the coefficient of friction. These were evidence that these students did not understand what quantity the slope and the area represented.

In the graphical problem in interview 4, finding the area meant accumulating the product of force and angle, which did not yield the total work. Six out of 9 students spontaneously stated that work equaled the area under the curve, but only one of them recognized the need for the radius factor without assistance from the interviewer. This was further evidence that although students could invoke the knowledge of "work equaled the area under the curve of force," they might not understand what quantity was being accumulated when calculating such an area. Therefore, they failed to apply that knowledge in novel situations.

4.4 Results from the fall 2009 study

4.4.1 Matching a definite integral with an area under the curve

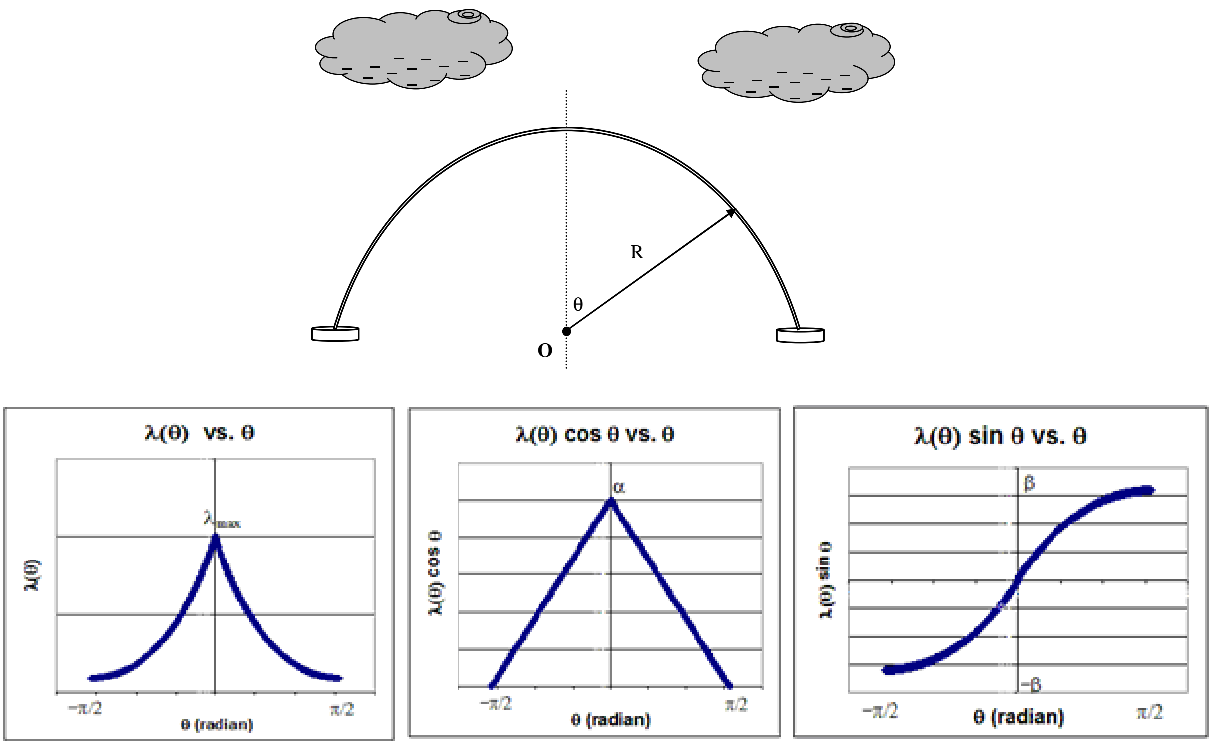
The graphical problems of interviews 5, 6, and 7 involved evaluating definite integrals using the area under the curve concept. All 15 students (S1 to S15) participating in these interviews solved the algebraic problems prior to the graphical problems. Each of these graphical problems provided three or four graphs describing the relation between the related quantities in the problem. Students had to select among these graphs the one in which the area under the curve was the value of the integral they encountered when solving the problem.

We found that most of the students preferred computing the integral algebraically to evaluating it graphically. Students attempted to find the algebraic expressions for the functions from the given graphs to plug into the integrals and computed them algebraically. Students considered evaluating the integrals using the area under the curve only when the integral was too complicated to be computed algebraically or when students were unable to find the explicit expressions for the functions. About half of the students in each interview were able to select the appropriate graph to find area (i.e. the graph of the integrand), while others needed hints on this task. The hint provided to the students in this situation was to draw a graph of an arbitrary function $f(x)$ and have students label the axes of the graph such that the area under the curve from $x = a$ to $x = b$ equaled the integral $\int_a^b f(x)\,dx$. This exercise, which directed students' attention to the relationship between the integrand and the function being plotted, helped most of the students recognize the correct graph to find the area.

4.4.1.1 Interview 5

Figure 4.4 The graphical problem in interview 5

You are standing at the center of the arch as in problem 1 in a stormy day. There are negatively charged clouds over the arch. The charge distribution on the arch now depends on the angle θ as per one of the graphs shown.



(a) Indicate the charge distribution on the figure below.
(b) Find the magnitude and direction of the electric field at your feet (i.e. at a point O on the ground directly below the top of the arch)

The graphical problem in interview 5 is presented in Figure 4.4. In this problem, students had to calculate the electric field due to a charged arch on which the charge distribution was a function of the angular position. According to Coulomb's law, the electric field due to the arch at its center was $E = \frac{1}{4\pi\varepsilon_0 R}\int_{-\pi/2}^{\pi/2} \lambda(\theta)\cos\theta d\theta$. Students were provided with the graphs of "$\lambda(\theta)$ vs. θ", "$\lambda(\theta)\sin\theta$ vs. θ", and "$\lambda(\theta)\cos\theta$ vs. θ", and had to evaluate the integral

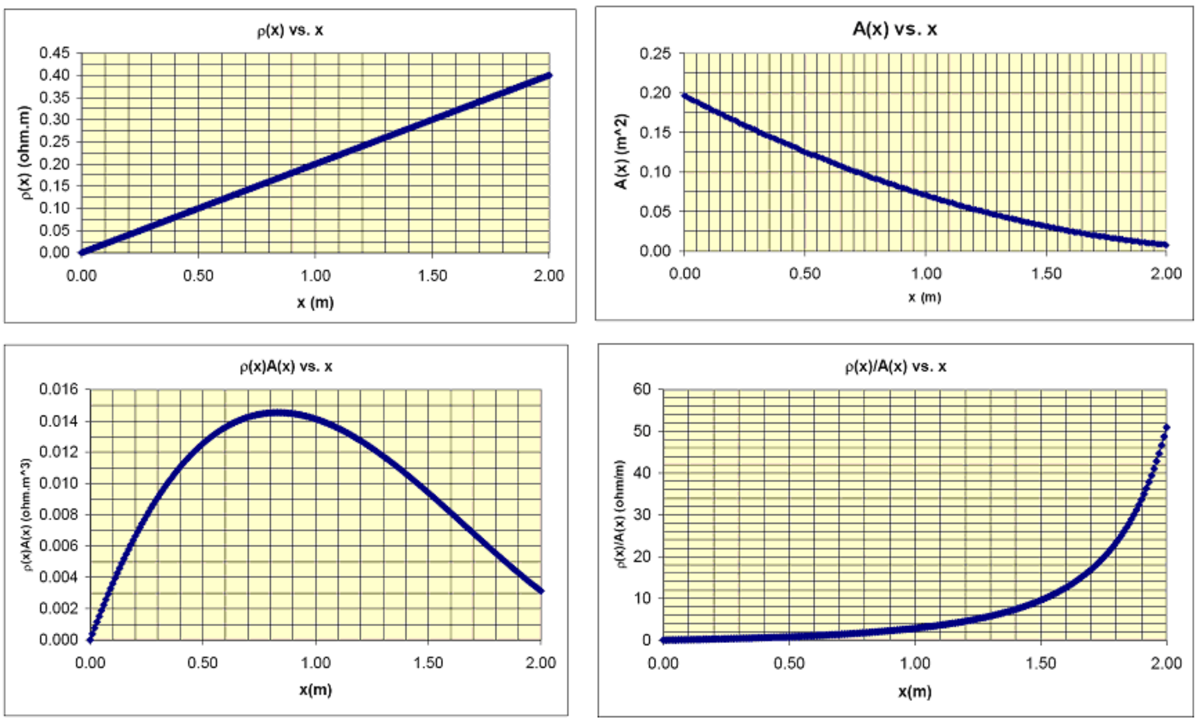
$\int_{-\pi/2}^{\pi/2} \lambda(\theta)\cos\theta d\theta$. The value of this integral equaled the area under the curve of "$\lambda(\theta)\cos\theta$ vs. θ" (i.e. the second graph in the problem statement) from $-\pi/2$ to $\pi/2$. One out of 15 students attempted to find the algebraic expression for $\lambda(\theta)$ to compute the integral algebraically. Four other students did not know what to do with the graphs. Upon being provided hints on the relation between a definite integral and an area under a curve, two of them were able to choose the correct graph to calculate the area under the curve, while the other two students needed further hints to recognize the correct graph. Out of the 10 students who spontaneously recognized the relation between the integral and the area under a curve, four students were able to choose the correct graph. The remaining six students initially chose the incorrect graph and needed hints to recognize the correct one. The errors these students made included: finding area under the curve of "$\lambda(\theta)$ vs. θ" (S11, S14, and S15) because they were "integrating $\lambda(\theta)$"; multiplying the area under the curve of "$\lambda(\theta)$ vs. θ" by $\cos\theta$ (S6); choosing the graph of "$\lambda(\theta)\sin\theta$ vs. θ" because "its area was easy to calculate" (S12); and relating the area with the anti-derivative of the integrand (S13).

4.4.1.2 Interview 6

The graphical problem in this interview is presented in Figure 4.5. This problem asked students to calculate the resistance of a conductor whose resistivity and diameter were changing along its length. The resistance of this resistor could be calculated by evaluating the integral $R = \int_0^2 \frac{\rho(x)dx}{A(x)}$, where $\rho(x)$ and $A(x)$ were the resistivity and the cross-sectional area of the conductor at position x. The graphs of "$\rho(x)$ vs. x", "$A(x)$ vs. x", "$\rho(x)\cdot A(x)$ vs. x", and "$\frac{\rho(x)}{A(x)}$ vs. x" were provided. Obviously, the value of the integral for the resistance equaled the area under the curve of "$\frac{\rho(x)}{A(x)}$ vs. x" (i.e. the fourth graph in the problem statement) from 0.0 m to 2.0 m.

Figure 4.5 The graphical problem in interview 6

A conductor has diameter decreasing from D to d over its length L. The resistivity of this conductor along the x axis is ρ*(x)* and its cross-sectional area is *A(x)*. The graphs of ρ*(x)* vs. *x*, *A(x)* vs. *x*, ρ*(x).A(x)* vs. *x*, and ρ*(x)/A(x)* vs. *x* are given. Find the resistance of this conductor.



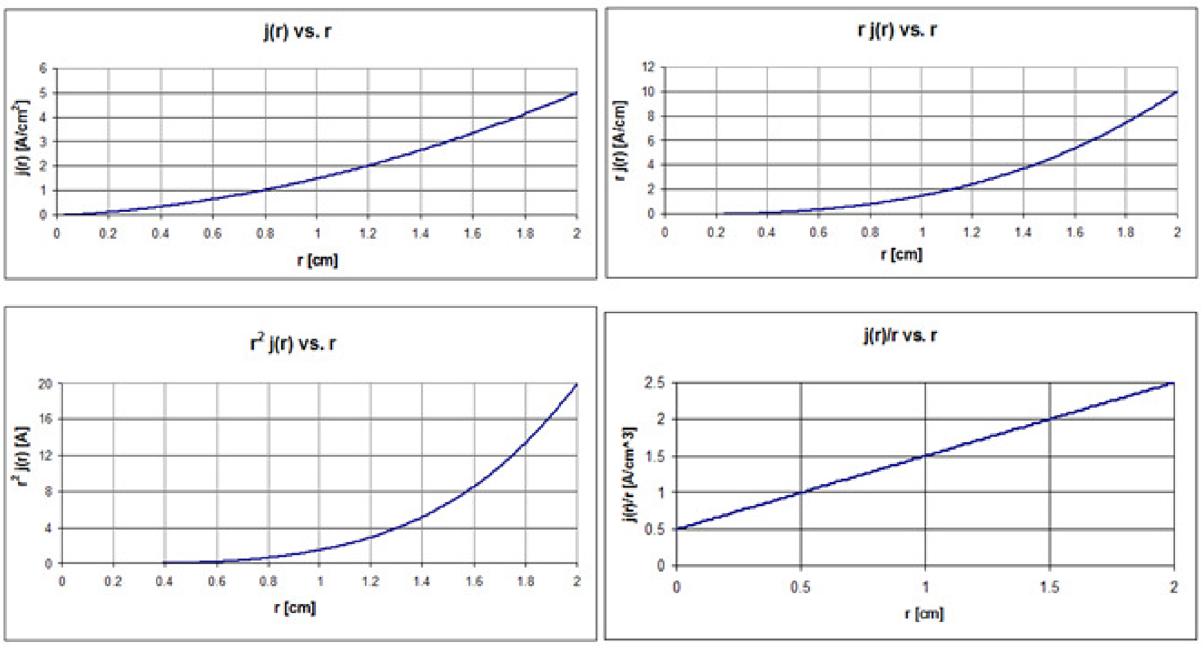
Three out of 15 students were able to choose the correct graph to evaluate the integral. Among the other 12 students, eight attempted to compute the integral algebraically by finding the algebraic expressions for $\rho(x)$, $A(x)$ and plugging in the integral. The expression for $\rho(x)$ could be easily obtained from the linear graph of "$\rho(x)$ vs. x", while the expression for the area function $A(x)$ had been derived in the algebraic problem which came before this problem. However, the obtained integral was too complicated for algebraic computation, so these 8 students considered evaluating the integral using area under the curve and all of them were able to pick the correct graph. The remaining four students were not able to choose the correct graph until being hinted by the interviewer. The errors these students made could be attributed to their misconceptions about basic properties of integral and the relationship between an integral and the area under a curve.

4.4.1.3 Interview 7

The graphical problem in interview 7 is presented in Figure 4.6. In this problem, students were asked to calculate the total current in a wire carrying a current with current density $j(r)$ changing across its cross-sectional area.

Figure 4.6 The graphical problem in interview 7

A cylindrical wire of radius R = 2 cm is carrying a current of density *j* which depends on the distance *r* from the center of the wire as per the graphs given. Find the magnitude of the magnetic field caused by the wire at a point P on its surface.



The equation for the current in this problem was $I = 2\pi \int_0^2 j(r)\, r dr$. Students were given the graphs of "$j(r)$ vs. r", "$rj(r)$ vs. r", "$r^2 j(r)$ vs. r", and "$\frac{j(r)}{r}$ vs. r". The value of the integral in the current equation equaled the area under the curve of "$rj(r)$ vs. r" (i.e. the second graph in the problem statement) from 0 cm to 2 cm. Nine out of 15 students were able to choose the correct graph. Four other students chose the "$j(r)$ vs. r" graph for the reason that the current density $j(r)$ was being integrated. The remaining two students chose the "$\frac{j(r)}{r}$ vs. r" graph because its area was easy to calculate.

In summary, almost all of the students indicated knowledge that an integral equaled the area under a curve. However, when provided with several graphs, students had difficulties identifying the graph on which the area under the curve was the value of a certain integral. There were four common errors that students made in selecting the graph:

- relating only one part of the integrand with the function being plotted (e.g. equating $\int_{-\pi/2}^{\pi/2} \lambda(\theta)\cos\theta d\theta$ with the area under the curve of "$\lambda(\theta)$ vs. θ", or $\int_0^2 j(r)rdr$ with area under the curve of "$j(r)$ vs. r");
- relating the area with the integrand (e.g. equating the area under the curve of "$\rho(x)$ vs. x" with the value of the function $\rho(x)$ in the integral $R = \int_0^2 \frac{\rho(x)dx}{A(x)}$);
- identifying the graph to find the area based on the simplicity of the area calculation (e.g. choosing a graph because the area calculation was straightforward);
- applying incorrect properties of integration (e.g. equating the integral of a quotient with the quotient of integrals).

In the next subsection, we will discuss students' misconceptions about integration and the area under a curve.

4.4.2 Students' misconceptions about the integral and the area under the curve

Our interviews also revealed some students' misconceptions about basic properties of integrals and the relationship between the integrals and the area under a curve. These misconceptions were the integral equals the area under the curve of the anti-derivative of the integrand, integral of a product or quotient equals sum or quotient of integrals, and integrand equals area under the curve. We will discuss each of these misconceptions below.

4.4.2.1 The area under a curve equaled the anti-derivative of the integrand

In the graphical problem in interview 5 (Figure 4.4), the integral was $\int_{-\pi/2}^{\pi/2} \lambda(\theta)\cos\theta d\theta$, which equaled the area under the curve of "$\lambda(\theta)\cos\theta$ vs. θ" from $-\pi/2$ to $\pi/2$. The following excerpt was taken from the interview with student S13 when she was attempting to choose the graph to find area.

Interviewer: *Okay, so now you have graphs …*

S13: *Yeah, I understand that I have to use these graphs, I just don't know how.*

Interviewer: *And you have your integral. So what is the relation between an integral and a graph?*

S13: *It's the area underneath the curve.*

Interviewer: *Uh huh, area under the curve. So which graph do you use to find the area?*

S13*: I'm hoping this one. [points at the graph of "* $\lambda(\theta)$ *vs.* θ"]

Interviewer: *Yes, you hope. But you should have a reason.*

S13: *No … It's this one [points at the graph of "* $\lambda(\theta)\sin\theta$ vs. θ"]

Interviewer: *How do you know you should use that graph?*

S13: *Um, because if I need the integral of cosine it's going to be sine so I need the area under this.*

This student was able to recognize that the integral equaled the "area underneath the curve" when hinted on the relationship between the integral and the graph. However, she was not sure which area was corresponding to the integral. After picking a graph with the "hope" that it would be the correct one, she was more thoughtful in her second attempt. Her explanation that the integral of cosine was sine indicated that she chose the graph based on the result of integrating the cosine in the integrand. This evidence suggested that she did not understand the relation "the integral equaled the area underneath the curve" although she could invoke it when solving the problem.

4.4.2.2 The area under a curve equaled the integrand

In the graphical problem in interview 6 (Figure 4.5), the integral was $\int_0^2 \frac{\rho(x)}{A(x)}dx$ which equaled the area under the curve of "$\frac{\rho(x)}{A(x)}$ vs. x" from 0.0 m to 2.0 m. Student S8 calculated the areas under the curves of "$\rho(x)$ vs. x" and "$A(x)$ vs. x" and plugged those areas into $\rho(x)$ and $A(x)$ in the integral. Similarly, in the graphical problem in interview 7 (Figure 4.6), student S3 calculated the area under the curve of "$j(r)$ vs. r" and plugged that area into $j(r)$ in the integral $\int_0^2 j(r)rdr$. These errors indicated that these students perceived the area under a curve as the value of the integrand rather than the value of the integral.

4.4.2.3 Integral of a product or quotient equaled a sum or quotient of integrals

In the graphical problem in interview 6 (Figure 4.5), student S1 found the explicit expression for $\rho(x)$ from the "$\rho(x)$ vs. x" graph and calculated the integral using the equation $\int_0^2 \frac{\rho(x)}{A(x)}dx = \int_0^2 \rho(x)dx + \int_0^2 \frac{dx}{A(x)}$. Students S6 and S8 attempted to use the equation $\int_0^2 \frac{\rho(x)}{A(x)}dx = \frac{\int_0^2 \rho(x)dx}{\int_0^2 A(x)dx}$ and calculated the quotient of the areas under the curves of "$\rho(x)$ vs. x" and "$A(x)$ vs. x".

4.4.3 Conclusions from the fall 2009 study

In summary, we found evidence that students might not completely understand the concept that "the integral equals the area under the curve" although they might be able to invoke it during problem solving. We also found evidence that some students held misconceptions about basic properties of integrals.

We answer our last research question – RQ3: To what extent did students understand the relationship between a definite integral and area under a curve?

Almost all of the students indicated knowledge of "the integral equaled the area under the curve," but only half of them (four students in interview 5, eight in interview 6, and nine in interview 7) were able to select the graph corresponding to a pre-determined integral when several graphs were present. The errors other students made – choosing a graph based on part of the integrand or on the simplicity of the area calculation – indicated that these students did not completely understand the relationship between a definite integral and area under a curve.

4.5 Discussion

In this study, we found that the majority of the students did not spontaneously invoke the area under the curve concept during physics problem solving. This might be attributed to students' unfamiliarity with the graphical methods as well as their strong inclination to algebraic methods in solving physics problems. Even when students invoked the area under the curve concept in a physics problem, there was evidence that they might not understand what physical quantity the area represented. We also found that when provided with several graphs, many students were unable to choose the graph on which the area under the curve equaled a pre-determined integral, even though they could state that the integral equaled the area under the curve.

We will now discuss how our findings support and extend other studies in mathematics and physics education research on students' use of the area under the curve concept.

Students' difficulties with the area under the curve concept in the physics context of our study are similar to those previously found in mathematics context. We found that most of the students used area under the curve to find work from a graph of force versus displacement but they might not understand why the work was equal to the area, so they failed to recognize that the area under the curve in the sphere problem was not yet the value of work. This is similar to what Artigue concluded in his study: most students could perform routine procedures of finding area under the curve but rarely could they explain why these procedures were necessary.

Thompson and Silverman (2006) suggested that for students to perceive the area under the curve as representing a quantity other than area (in our case it was work), students must be able to see the integration process as an accumulation of the incremental bits that were formed multiplicatively. The hints we provided to help students recognize the use of the area under the curve concept in our interviews aimed at this goal. We asked students questions that directed their attention to the fact that the total work was the accumulation of the product of force and distance over small increments, which was essentially the area under the curve on the graph of force versus linear displacement.

Sealey (2006) concluded that the area under the curve method could be a powerful tool to evaluate a definite integral only when students understood the structure of the definite integral. Our study showed the extent to which students struggled with choosing an area that equaled a definite integral when they did not view the integral as having two components: the integrand and the infinitesimal term dx or dr. About half of the students in our interviews chose the incorrect graph because their choice was based on the wrong clues (i.e. based on part of the integrand, the anti-derivative of the integrand, or the ease of finding the area). The hints that asked students to label a graph of an arbitrary function $f(x)$ such that the area under the curve equaled the integral $\int_a^b f(x)\,dx$ directed students' attention to the two components of an integral and helped them recognize that the integrand was the clue for choosing the correct graph.

McDermott *et al.* (1986) studied how students used area under the curve in kinematics. Our study investigated students' use of area under the curve in many other topics of introductory physics. We did not have any problems involving negative area as in McDermott *et al.*'s study, but we had problems with more than one graph from which we could investigate how students related a definite integral with an area under a curve.

4.6 Limitations and future work

The research methodology used in this study was individual interview. This method had an advantage that it allowed us to gain insight into how individual students interacted with the concept of an integral as area under the curve. It also allowed us to interview the same students several times during two semesters, and therefore, we could track the development of a student through the courses. In spite of the advantages afforded by individual interviews, the method limited the number of students participating in the study, and hence, limited the generalizability of the results.

Our interview problems involved several physics quantities that could be calculated using area under the curve. However, there was no problem involving negative areas or areas that had the lower bound other than the $f(x)=0$ axis (i.e. the x-axis). By "area under the curve" we usually mean the area bounded by the curve and the $f(x)=0$ axis. There are problems in which the "area under the curve" is bounded by the curve and the $f(x)=-2$ line for instance. Investigating whether students know "integral equals area under the curve, but above what?" will be an interesting study following the study presented in this paper.

Based on our interview findings, we plan to develop tutorial materials to help students understand the "integral equals area under the curve" relationship and implement them for all of the students in both EP1 and EP2 courses (usually around 200+ students each) in the future semesters when the courses are offered to test the effects of those materials in helping students learn to use the area under the curve method in physics problem solving.

Chapter 5 - Tutorials to facilitate students' application of the integral and the area under the curve concepts in work – energy problems

5.1 Motivation and Introduction

In the spring 2009 study, we found that students in introductory mechanics encountered significant difficulties in applying the integral and the area under the curve concepts to mechanics problems. The major difficulties included not recognizing the use of these two concepts in the problems, and not understanding the accumulation process when doing the integral or finding an area under the curve. Many of these students, however, were eventually able to solve those problems with verbal hints provided by the facilitator. This suggested that the students would have been able to apply the integral and the area under the curve concepts in mechanics problems if they had received appropriate scaffolding which targeted their difficulties. In other words, these problems were well within these students' Zone of Proximal Development (Vygotsky, 1978)

Based on the knowledge of the difficulties that students encountered and the scaffolding that might be helpful, we developed and tested instructional materials, which will be referred to as tutorials, to facilitate students' application of the integral and the area under the curve concepts in mechanics problems. Each tutorial had two components:

- a set of exercises created to help students learn the knowledge and skills necessary to enable them to apply the integral and the area under the curve concepts in physics problems;
- a protocol for the conversation between the facilitator and the students to facilitate students' construction of ideas as they worked through the set of exercises.

In the spring 2010 semester, we created and tested four tutorials on different topics of introductory mechanics as follows:

- Tutorial 1: One-dimensional kinematics
- Tutorial 2: Newton's laws and forces
- Tutorial 3: Work – energy for a point mass
- Tutorial 4: Work – energy for a rigid body

The first two tutorials focused on helping students build the skills to apply the derivative concept and vector addition to physics problems. In this chapter, we will only discuss the tutorials 3 and 4 because they aimed at helping students learn to apply the integral and the area under the curve concepts in physics problems, and they also had similarly structured sets of exercises. These sets of exercises consisted of one or two pairs of matched math and physics exercises, a debate problem, and two problem posing tasks. We tested the effect of our tutorials in comparison with standard instructional materials on problem solving. In this study, we defined "standard instructional materials on problem solving" (or "standard instructional materials" in brief) as the practice of providing students with sample problems and written solutions after students had attempted the problems themselves.

We will present the rationale of the tutorials 3 and 4, and their impact on students' ability to apply the integral and the area under the curve concepts in physics problems on work – energy. The research question for this study is: To what extent did our tutorials help students improve their ability to apply the integral and the area under the curve concepts in work – energy problems, compared to standard instruction (i.e. sample problems and solutions)?

5.2 Rationale of the tutorials and the standard materials

In this section, we will present the rationale for the creation of the exercises of the tutorials 3 and 4, and the selection of the sample problems and solutions to be used to represent standard instructional materials in our study.

The purpose of the tutorials was to help students learn to apply mathematical concepts (i.e. the integrals and the area under the curve) to physics problems. This task required students to invoke mathematical knowledge or model and then apply it to a physical context. We found from our study in the spring 2009 that students had a lot of difficulties in doing such a task, although the mathematical models and the physics knowledge required in our interview problems were very familiar for most students (i.e. the integral and the area under the curve; work-kinetic energy theorem). We suspected that the physics context might hinder the mathematical model which made students fail to recognize the application of the mathematical model in the physics problems. So our ideas for creating the tutorial were to provide students with an intermediate step in applying a mathematical model to physics problems. Specifically, we provided students with a simple math exercise in which they only needed to recall a familiar math model. This intermediate step offered students an opportunity to invoke, in a context-free environment, the math model necessary to be applied to the physics exercise ahead. Then came the physics exercise in which students applied the math model in the previous step to a simple physical context. This strategy made it clear to students how a mathematical model could be applied to a physical situation. So part of the exercise set of our tutorials was a sequence of matched (related) math and physics exercises. In this sequence, a mathematical model was invoked in the math exercise and then was applied to the physics exercise that followed. This sequence, therefore, suited well with the vertical and horizontal framework mentioned in the literature review.

Besides preparing students with the ability to apply mathematical models to physics scenarios, our tutorials also aimed at helping students prepare the physics knowledge necessary to solve complete physics problems, in which the mathematical models were applied. We found from the spring 2009 study that students also had difficulties applying basic physical principles (e.g. conservation of energy, work-kinetic energy theorem) to the interview problems. So in each of our tutorials, there was a debate problem in which students were asked to comment on the strategies suggested by fictitious students for solving a physics exercise. The strategies that these fictitious students suggested contained errors that we observed our students made in the spring 2009 interviews. By reflecting on other students' errors, students doing the debate problem might be able to avoid those errors in their own solutions when they solve similar problems.

The last exercise in each of our tutorials was a problem posing task. Students were asked to create a problem of their own using the mathematical model and the physics scenarios of the previous exercises, and to write an instruction for solving the problem they created. The purpose of this task was to help students learn to integrate the mathematical model with a physical context. Table 5.1 below summarizes the types of exercises in our tutorial and their purposes.

Table 5.1 The types of exercise in the tutorial and their purposes

Type of exercise		Purpose
Sequences of	Math exercise	Helps students build mathematical model in a context-free environment
	Physics exercise	Helps students apply the mathematical model in the math exercise to a simple physics context.
Debate problem		Prepares students with the necessary physics background to solve a complete physics exercise.
Problem posing task		Helps students learn how a mathematical model can be applied to a physics exercise.

After completing each of the exercises, students were asked to check with a facilitator before proceeding to the next problem. The protocols for the conversation between the facilitator and the students after each exercise of these tutorials were very similar. The facilitator first asked students to explain what they had done and then checked the correctness of their solution to the exercise. If the students did the exercise correctly, the facilitator would then ask students about what ideas they had learned from doing the exercise and how those ideas might help them solve similar exercises in the future. If the students did not get the correct answer to the exercise, the interviewer would provide hints to help students recognize and correct their errors.

The criteria for the selection of the standard instructional materials were that they were similar to typical end-of-chapter problems in terms of their structure and were similar to the problems in the tutorial in terms of physics concepts and representations. Specifically, the standard materials adopted the mathematic and physics concepts and representations from the tutorial but must not contain math exercises, debate problem, and problem posing tasks, because these types of problems were unlikely to appear in typical introductory physics textbooks. These types of exercises should be replaced by equivalent physics exercises in the standard materials. Another criterion was that the amount of time to complete one set of the standard material must be equivalent to the amount of time needed to complete one tutorial. This criterion was to ensure equivalent amount of practice students taking each set of material experienced. Table 5.2 below summarizes the changes that were made on the tutorial when the standard material was created.

Table 5.2 Comparison of the types of exercises in the tutorial and the standard material

Type of exercise in tutorial		**Type of exercise in standard material**
Sequence of math and physics exercises	is replaced by	All physics exercises similar to the physics exercises in the tutorial
Debate problem		Physics exercise discussed in the debate problem
Problem posing tasks		Physics exercises

5.2.1 Tutorial 3

5.2.1.1 Creation of treatment group materials for tutorial 3

The exercise set of tutorial 3 consisted of two pairs of matched math and physics exercises (one pair in algebraic representation and the other in graphical representation), a debate problem, and two problem posing tasks. The pairs of math and physics exercises were to teach students about the accumulation process underlying the integral and the area under the curve. Each pair of exercises was expected to help students recall the necessary mathematical knowledge in a context-free math exercise and then applied that knowledge to a physical situation, i.e. the physics exercise. The debate problem was intended to prepare students with the physics background needed to do problems involving work and energy. The problem posing task provided students with an opportunity to practice putting together the knowledge on the integral and the area under the curve concepts with the physics background to create and solve complete problems involving the integral and the area under the curve.

The topic of the tutorial 3 was work – energy of a point mass, which was the same as the topic of the interview 3 in the spring 2009 study. So the exercise set of tutorial 3 was created based on the findings about students' difficulties and the helpful hints in interview 3 of the spring 2009 study. The interview 3 in the spring 2009 study involved finding the work done by a non-constant force using the integral and the area under the curve concepts. We found that only 3 out of 11 students in the A-G sequence and 3 out of 9 students in the G-A sequence could spontaneously recognize that the work equaled the integral of force or the area under the curve of force, respectively. Other students attempted to use the formulas for the work done by a constant force or kinetic friction force on a horizontal floor, i.e. $W = F.d$ and $W = \mu mgd$, to calculate the work done by the resistance force of the liquid. Five students in the A-G sequence and 3 students in the G-A sequence were able to recognize that work equaled the integral and the area under the curve after hints were provided by the interviewer. The hint was to guide students' thinking about the total work as the sum of the infinitesimal works on small segments of the path, i.e. thinking about integrating a function and finding an area under the curve as an accumulation process. Since this hint had proven to be effective, we employed its idea in creating the exercise set for the tutorial 3. So the goal of the exercise set in tutorial 3 would be to help students learn to view the integral and the area under the curve as an accumulation.

Exercise 1 (Figure 5.1) asked students to calculate the integral $\int_a^c f(x)dx$ given the graph of $f(x)$ vs. x. This could be done by finding the area under the curve of $f(x)$ vs. x from $x = a$ to $x = c$, which equaled the sum of the areas of a rectangle and a trapezoid. This simple math problem was an example of accumulation in mathematics. It might help students recall that the integral notation $\int_a^c f(x)dx$ represented the sum of the product of $f(x)$ and dx at every value of x from $x = a$ to $x = c$. The product of $f(x)$ and dx was actually the area of a trapezoid of height $f(x)$ and width dx. So the integral $\int_a^c f(x)dx$ represented the sum of the area of all trapezoids, which essentially equaled the total area under the curve of $f(x)$ vs. x from $x = a$ to $x = c$.

Exercise 2 (Figure 5.2) was an application of the mathematical idea in exercise 1 to a physical situation. In this exercise, students had to calculate the work done by a force $F(x)$ over a distance d. The magnitude of the force was not constant over the distance d and was given by the graph of $F(x)$ vs. x. Students had learned that work equaled force time distance, i.e. $W = F \cdot d$, which was applicable only when the force F was constant over the whole distance d. So when a non-constant force was presented, the total work must be calculated by adding all of the works on small segments of the distance, over which the force could be considered constant. The work on each segment was the product of the force $F(x)$ and the length dx of a small segment of the path, which was actually the area of a trapezoid under the curve of force. So the total work on the whole distance d would then equal the total area under the curve of $F(x)$ vs. x from $x=0$ to $x=d$.

Figure 5.1 Exercise 1 of the tutorial 3

Exercise 1 – Tutorial 3

The graph of a function $f(x)$ is given below.

Find the value of the integral $\int_a^c f(x)\,dx$ in terms of the constants *a*, *b*, *c*, *m*, *n*.

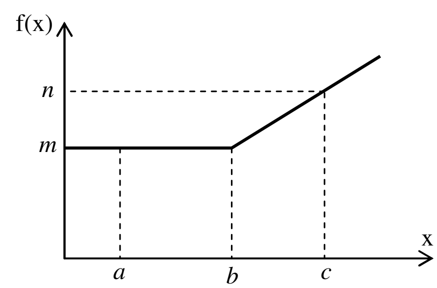
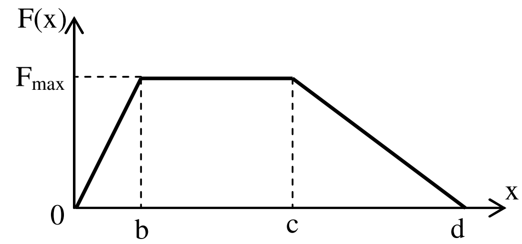


Figure 5.2 Exercise 2 of the tutorial 3

Exercise 2 – Tutorial 3

The graph below shows the magnitude of a force F(x) acting on an object with respect to the displacement x of the object (F is in Newton and x is in meter). Find the work done by force F on the object over the distance d that the force is acting.



Exercises 3 and 4 (Figure 5.3 and Figure 5.4) formed another pair of matched math and physics exercises. Exercise 3 asked students to calculate the area limited by the curves of some functions. This area could be calculated by adding up the areas of all thin trapezoids under the curve, which was essentially the value of the integral. So the exercise 3 was the inverse of the exercise 1. In exercise 1, students calculated an integral using the area under the curve, while in exercise 3, students calculated an area under the curve using the integral. Exercise 3, therefore, reinforced the idea of how the integral was related to the area under the curve: they both represented the accumulation of small quantities to obtain the total quantity.

Similarly, exercise 4 was the inverse of exercise 2. In exercise 4, students had to calculate the work done by a non-constant force over a distance from the function of force. This work could be calculated by adding up the works on all small segments of the distance, i.e. adding up the product of $F(x)$ and dx, which was essentially the value of the integral of force. The exercise 4, therefore, also reinforced the idea of how the work could be calculated by integrating the force function.

Figure 5.3 Exercise 3 of the tutorial 3

Exercise 3 – Tutorial 3

Find the area of the region surrounded by the graphs of the following functions:
$f(x) = x^3 + 2x + 1,\ f(x) = 0,\ x = x_1,\ x = x_2$.

Figure 5.4 Exercise 4 of the tutorial 3

Exercise 4 – Tutorial 3

A block is pulled on a horizontal frictionless floor by a force F whose magnitude (in Newton) depends on the displacement x of the block (in meters) as per the function: $F(x) = ax^2 + bx + c$ (a, b, c are constants). Find the work done by force F when the block has been moved from x_1 to x_2.

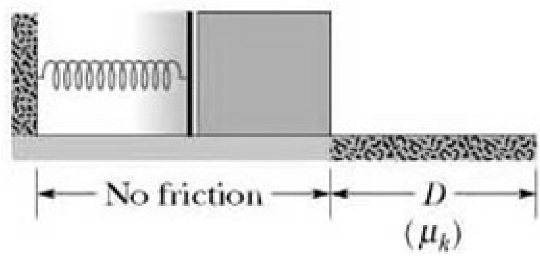
Exercise 5 (Figure 5.5) was a debate problem, in which students were given a conversation of several fictitious students discussing about their strategies to solve a physics problem. The discussion focused on how to apply the conservation of energy principle, and how to calculate the work done by friction in the problem. The reasoning of some fictitious students was correct, while others' was not. The errors that the fictitious students made in their reasoning were the common errors observed in our interviews in the spring 2009 study. Students were asked to comment on the reasoning of each of the fictitious students and to indicate the fictitious students who had correct strategies. By reflecting on other students' errors, our students were expected to be able to avoid those errors when solving problems similar to the problem being discussed in the debate problem.

Figure 5.5 Exercise 5 of the tutorial 3

Exercise 5 – Tutorial 3

Five students are discussing their strategies to solve the following problems.

A 3.5 kg block is accelerated from rest by a spring, spring constant 632 N/m that was compressed by an amount x. After the block leaves the spring it travels over a horizontal floor with a coefficient of kinetic friction $\mu_k = 0.25$. The frictional force stops the block in distance D = 7.8 m.



What was the spring compression *x*?

Below are parts of the students' strategies. Comment on each student's ideas. Explain who you agree with most and why. For the students who make statements you disagree with, try to identify what went wrong in the student's reasoning.

Student	Strategy	Comments
David	Energy is conserved so all the changes in energy add to zero. The block starts from rest and then comes to a stop, so there is no change in kinetic energy. The only energy that changes is the spring's potential energy and that's good because that involves the compression of the spring. You can calculate the change in potential energy and solve for the compression.	
Mary	Friction is involved so you need to use $\Delta K + \Delta U = W$, where $W = -\mu_k mgD$ is the work done by friction. ΔK is zero because initial and final speeds are zero. The initial U is that of the spring and final U is zero. Then put everything into the equation and solve for x.	
Eric	Isn't the work $+\mu_k mgD$, because W in that equation is the amount of work done and therefore it must be positive?	
Susan	But the spring does work on the block too and you have to take that into account. Work is force times distance, and since the force of the spring is -kx and the spring pushes the block a distance x, the work done by the spring is $-kx^2$. That's the formula you should use to find the compression.	
Mike	All you have to do to calculate the work done by the spring is to plug in the total distance the spring pushes the block into the force -kx. So, if the initial compression is L, the work done by the spring is -kL.	

Exercise 6 (Figure 5.6) consisted of two problem posing tasks. Each task asked students to create a solvable problem of their own in which the physics scenario of the problem discussed

in exercise 5 and the physics ideas in exercises 2 or 4 were employed. Students were also asked to write an instruction to solve the problems they created. This exercise was expected to help students learn to embed an integration task into a physics scenario to make a complete physics problem involving integration.

Figure 5.6 Exercise 6 of the tutorial 3

Exercise 6 – Tutorial 3

a. Start with the physics problem in problem 5, modify it by including in it the physics ideas in problem 2 to create a new solvable problem of your own. Write your instructions to solve that new problem.

b. Start with the physics problem in problem 5, modify it by including in it the physics ideas in problem 4 to create a new solvable problem of your own. Write your instructions to solve that new problem.

5.2.1.2 Creation of control group materials for tutorial 3

As mentioned above, the criteria for creating the materials for the control group were that these materials represented typical end-of-chapter exercises and were similar to the exercises in the tutorials in terms of the physics concepts and representations. So we started with our tutorial exercises and select textbook-like exercises that covered the same concepts and had the same representations.

It is unlikely that a textbook in introductory physics prepares students with the mathematical knowledge necessary for an exercise before introducing the exercise. So the first difference between the standard material and our tutorial was that the standard materials did not contain math exercises. For this reason, the exercises 1 and 3 in the tutorials (which were math exercises) were replaced by equivalent physics exercises in the standard material. The exercises 1 and 3 in the standard material were physics exercises in which students were asked to find the work done by non-constant forces. The exercises 2 and 4 in the standard material were also physics exercises on the work done by non-constant force, but with numerical values instead of algebraic variables as in exercises 1 and 3.

Typical physics textbooks do not contain debate problems and problem posing tasks. So the problems 5 and 6 in the tutorial (which were the debate problem and the problem posing tasks, respectively) were also removed in the standard material. Instead, exercise 5 in the standard material was the exercise being discussed in the debate problem in the tutorial. Students using the standard material solved the exercise and referred to its solution upon completing, rather than judging other students' reasoning about the strategies to solve the exercise without actually solving it.

We estimated that the time it took to complete the exercises 1, 3, and 5 in the standard material was longer than the time it took to do the corresponding exercises in the tutorial. The amount of time to solve the problem posing task in the tutorial might compensate for this difference. This was the reason that there were only 5 exercises in the standard material compared to 6 exercises in the tutorial, but equivalent total time on task was ensured.

All problems in the standard material 3 are presented in Figure 5.7 to Figure 5.11 below.

Figure 5.7 Exercise 1 of the standard material 3

Exercise 1 – Standard material 3

The graph below shows the magnitude of a force F(x) acting on an object with respect to the displacement x of the object (F is in Newton and x is in meter). Find the work done by force F on the object over the distance d that the force is acting.

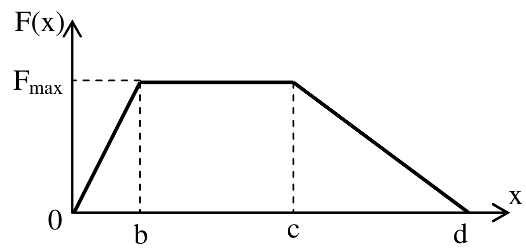


Figure 5.8 Exercise 2 of the standard material 3

Exercise 2 – Standard material 3

The graph below shows the magnitude of a force F (in Newton) acting on an object with respect to the displacement x (in meters) of the object. Find the work done by force F on the object over the displacement from 0 m to 10 m.

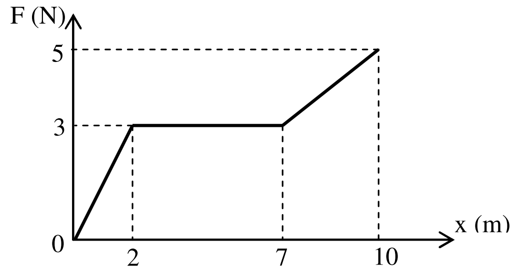


Figure 5.9 Exercise 3 of the standard material 3

Exercise 3 – Standard material 3

A block is pulled on a horizontal frictionless floor by a force F whose magnitude (in Newton) depends on the displacement x of the block (in meters) as per the function: $F(x) = ax^2 + bx + c$ (a, b, c are constants). Find the work done by force F when the block has been moved from x_1 to x_2.

Figure 5.10 Exercise 4 of the standard material 3

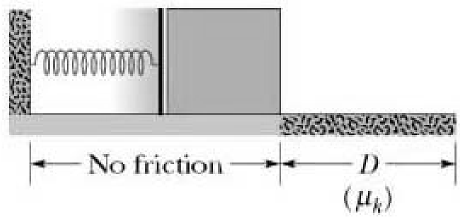
Exercise 4 – Standard material 3

A block is pulled on a horizontal frictionless floor by a force F whose magnitude depends on the displacement of the block as per the function: $F(x) = 2x^3 - 3x + 2$ (x is in meter, F is in Newton). Find the work done by force F when the block has been moved from 0 m to 2 m.

Figure 5.11 Exercise 5 of the standard material 3

Exercise 5 – Standard material 3

A 3.5 kg block is accelerated from rest by a spring, spring constant 632 N/m that was compressed by an amount x. After the block leaves the spring it travels over a horizontal floor with a coefficient of kinetic friction μ_k = 0.25. The frictional force stops the block in distance D = 7.8 m.



5.2.2 Tutorial 4

5.2.2.1 Creation of treatment group materials for tutorial 4

Tutorial 4 consisted of three pairs of matched math and physics exercises, a debate problem, and two problem posing tasks. The first pair of math and physics exercises was to remind students about the relationship between the distance along a circle and the angle it spanned at the center of the circle. The other two pairs of math and physics exercises were to familiarize students with converting the variable of a function to get the function in the desired variable, and to calculate a physical quantity from the new function. The debate problem was intended to prepare students with the physics background needed to solve problems involving work and energy of a rigid body. The problem posing tasks provided students with an opportunity to integrate the physics context and the mathematical tools.

The topic of the tutorial 4 was work – energy of a rigid body, which was the same as the topic of the interview 4 in the spring 2009 study. So the exercise set of tutorial 4 was created based on the findings about students' difficulties and the helpful hints in interview 4 of the spring 2009 study. In this interview, students had to calculate the work done by the rolling friction force between a sphere and a circular track. The force was given as a function of the angular displacement of the sphere on the track in algebraic and graphical representations. We found that the major difficulty students encountered in this interview was not recognizing that the integral of force or the area under the curve in this interview was not the value of work, because the force was given as a function of the angular displacement. The correct value of work must be calculated using the integral or the area under the curve of force versus linear displacement. So prior to doing the integral or finding the area under the curve, students must convert the variable to have the given force as a function of linear displacement. The variable conversion from angular displacement θ to the linear displacement s could be done using the relationship $\theta = \frac{s}{R}$ in which R is the radius of the track.

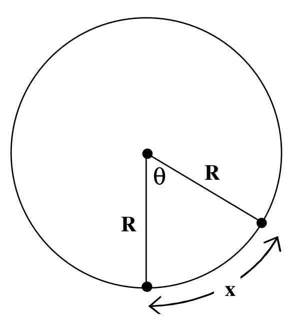
The exercise set of the tutorial 4 aimed at helping students do the variable conversion on a function. The purpose of exercise 1 (Figure 5.12) was to help students recall the relation between an angle and the distance it spanned on a circular track. Part A of this exercise was a math question on the relationship between the angle θ and the length x it spanned on the edge of a circular disk of radius R. Part B of exercise 1 was a physics question in which this relationship was employed.

Figure 5.12 Exercise 1 of the tutorial

Exercise 1 – Tutorial 4

a. What is the length of the arc 'x' along a circle in terms of radius R and angle θ (in radian)?

b. A bug sits on the edge of the turn table of radius R = 2.0 m which is rotating around its center. What is the distance 'x' that the bug has traveled after the turn table has rotated by an angle θ = π/4 ?



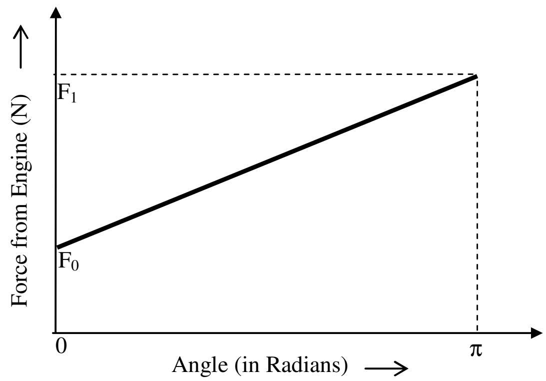
Exercise 2 (Figure 5.13) was intended to teach students to find the work done by a force when the force was given as a graph of force versus angular displacement. Part A of this exercise

was a math question which asked students to convert the graph of a function to the graph of the same function with respect to a different variable. Specifically, students were asked to convert the graph of force versus angular displacement to the graph of the same force versus linear displacement. Using the relation $s = R\theta$, the values on the horizontal axis of the graph could be converted into linear displacement, while the values on the vertical axis (i.e. the magnitude of force) remained unchanged. Part b of exercise 2 was a physics question which asked students to calculate the work done by the force in part a. Once the graph of force versus linear displacement had been obtained, the work could be calculated by simply finding the area under the curve of force on this graph. This exercise was aimed to help students reflect on two graphs of the same function, and how the work could be calculated using one graph but not the other.

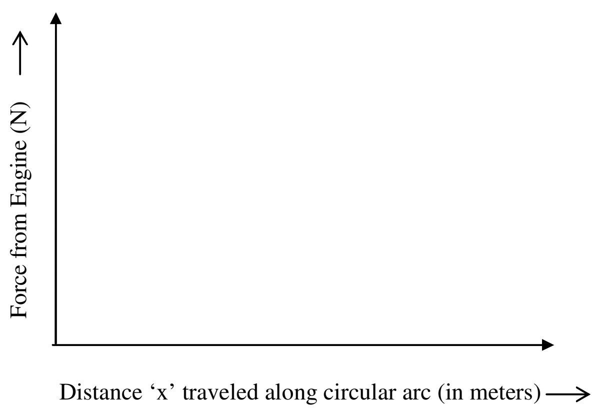
Figure 5.13 Exercise 2 of the tutorial 4

Exercise 2 – Tutorial 4

A toy plane is attached to a pole by a string and flies around it in a circular arc of radius R (in meters). The graph below shows the force exerted by the engine of the plane as it starts from rest from its initial position ($\theta = 0$ radian) to the final position ($\theta = \pi$ radians).



a. Plot the graph of force of the engine (in Newton) with respect to the distance 'x' (in meter) that the plane travels along the circular arc from its initial to its final point.



b. Find the work done by engine when the plane travels from its initial point to the final point.

Similarly, exercise 3 (Figure 5.14) had the same purpose and procedure as exercise 2, but with algebraic representation of the force function. Part a of exercise 3 was a math question which asked students to convert the variable of a function. Specifically, the given function was a function of the angular displacement as the variable, and students had to convert it to a function of linear displacement. This could be accomplished by replacing the angular displacement θ in the given function with the linear displacement s using the relation $\theta = \frac{s}{R}$. Part B of exercise 3 was a physics question which asked students to calculate the work done by the a force, given the force function in part a. Once the function of force with respect to linear displacement had been obtained, the work could be calculated by integrating this function with respect to its variable. This problem helped students reflect on how a function could be written with respect to different variables, and how the work could be calculated by integrating the function with respect to one variable but not the others.

Figure 5.14 Exercise 3 of the tutorial 4

Exercise 3 – Tutorial 4

A toy plane is attached to a pole by a string and flies around it in a circular arc of radius R = 3.0 m. The equation below shows the force exerted by the engine of the plane as it starts from rest from its initial position (θ = 0 radian) to the final position (θ = π radians)

$$F(\theta) = \mathrm{a}\theta + \mathrm{b}$$

where, a, b are constants; F is in Newton, and θ is in radian.

a. Write down the equation of force of the engine as a function of the distance ‘x’ the plane travels along the circular arc from its initial to its final point.

b. Find the work done by the engine when the plane travels from its initial point to the final point in terms of a and b.

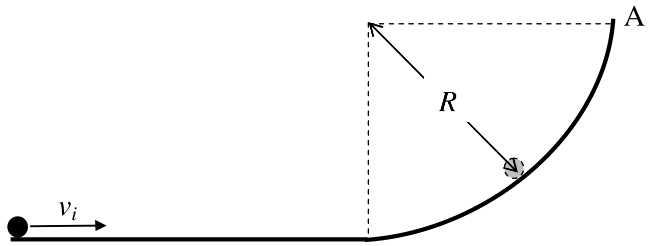
The debate problem of the tutorial 4 (Figure 5.15) was intended to prepare students with the physics background to solve problems involving work – energy of a rigid body. Students were provided a problem which asked for the speed of a hoop as it left the circular track and the discussion of 5 fictitious students on how to solve the problem. The reasoning of some of the fictitious students was correct, while others’ reasoning was not. The errors that these students made were the common errors that we found in our interview 4 in the spring 2009 study. Our students were then asked to comment on the reasoning of each of the fictitious students. By reflecting on other students’ mistakes, our students might be able to avoid them in their own solutions.

Figure 5.15 Exercise 4 of the tutorial 4

Exercise 4 – Tutorial 4

Five students are discussing their strategies to solve the following problem.

A hoop radius r = 1 cm and mass m = 2 kg is rolling at an initial speed v_i of 10 m/s along a track as shown. It hits a curved section (radius R = 2.0 m) and is launched vertically at point A.



What is the launch speed of the hoop as it leaves the slope at point A?

Below are parts of the students' strategies. They may not be the complete solutions. Comment on each student's ideas. Explain who you agree with most and why. For the students who make statements you disagree with, explain what you think is wrong in the student's reasoning.

	Strategy	Comments
David	Energy of the hoop is conserved. On the straight part of the track, the hoop's energy includes both translational and rotational kinetic energy. At point A, the hoop's energy includes potential and translational kinetic energy. When the hoop flies off the track, it does not roll any more, so it does not have rotational kinetic energy at point A.	
Mary	Yes, the hoop does not have rotational energy at point A, but it does not have translational energy on the straight part of the track either. The hoop doesn't have translational motion. It moves forward because it is rolling along the track.	
Eric	The hoop has both translational and rotational motion both on the straight part of the track and at point A. So there are two kinds of kinetic energy in both initial and final energy.	
Susan	Both gravity and normal forces, which are acting on the sphere, do not cause any torque to the sphere so angular momentum of the sphere is conserved between initial point and point A. Angular momentum equals to moment of inertia times angular speed, so I can find angular speed at point A. This angular speed divided by the radius of the sphere is the linear speed of the sphere at point A.	
Jim	I will use kinematics equation: $v^2 = v_0^2 + 2ad$, where a is acceleration due to gravity which is acting on the sphere as it climbs up the track and d is the distance along the track. Then I can find speed of the sphere at point	

Exercise 5 of tutorial 4 (Figure 5.16) consisted of two problem posing tasks, which asked students to create their own problems by combining the physics principles of the physics problem being discussed in the debate problem with the algebraic and graphical representations of the force function discussed in exercises 2 and 3. This exercise was intended to help students learn to embed a unit conversion task and an integration task into a physics scenario to make a complete physics problem.

Figure 5.16 Exercise 5 of the tutorial 4

Exercise 5 – Tutorial 4
a. Start with the physics problem in problem 4, modify it by including in it the physics ideas in problem 2 to create a new solvable problem of your own. Write your instructions to solve that new problem.
b. Start with the physics problem in problem 4, modify it by including in it the physics ideas in problem 3 to create a new solvable problem of your own. Write your instructions to solve that new problem.

5.2.2.1 Creation of control group material for tutorial 4

The criteria for creating the material for the control group for tutorial 3 also applied for the creation of the control group materials for tutorial 4. We removed all of the math exercises, debate problem, and problem posing tasks from the tutorial 4 and put in physics exercises covering the same concepts and had the same representations to the standard material. Specifically, part a of exercise 1 in the tutorial was a math question, so it was removed in the standard material. Exercises 2 and 3 in the standard material 4 were adopted from the corresponding exercises in the tutorial 4, but with the math questions (parts a of these exercises) removed.

The debate problem in the tutorial 4 was replaced by a physics exercise in the standard material 4. This was the physics exercise that was discussed by the fictitious students in the tutorial 4. The problem posing tasks were also removed in the standard material 4.

Due to the removal of the math questions and the problem posing tasks from the tutorial 4, the standard material 4 was shorter and seemed to take less time to complete than the tutorial 4. So to ensure the equivalent amount of practice time, we added one more physics problem to the standard material 4, which was the exercise 5 in this material.

All exercises in the standard material 4 are presented in the Figure 5.17 to Figure 5.21.

Figure 5.17 Exercise 1 of the standard material 4

Exercise 1 – Standard material 4

A bug sits on the edge of the turn table of radius R = 2.0 m which is rotating around its center. What is the distance 'x' that the bug has traveled after the turn table has rotated by an angle $\theta = \pi/4$?

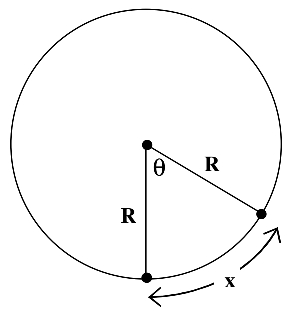


Figure 5.18 Exercise 2 of the standard material 4

Exercise 2 – Standard material 4

A toy plane is attached to a pole by a string and flies around it in a circular arc of radius R = 3.0 m. The graph below shows the force exerted by the engine of the plane as it starts from rest from its initial position ($\theta = 0$ radian) to the final position ($\theta = \pi$ radians).

Find the work done by the engine when the plane travels from its initial point to the final point.

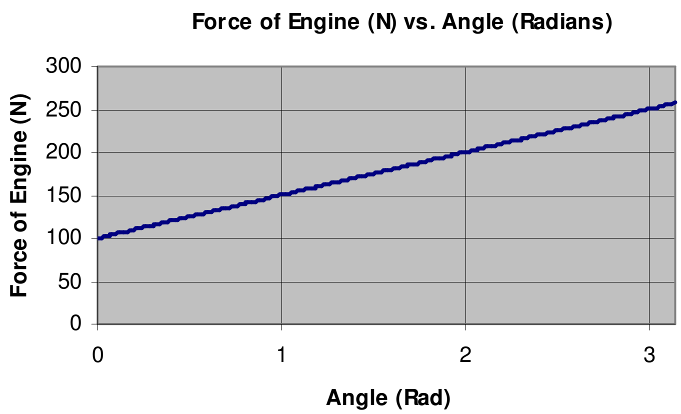


Figure 5.19 Exercise 3 of the standard material 4

Exercise 3 – Standard material 4

A toy plane is attached to a pole by a string and flies around it in a circular arc of radius R = 3.0 m. The equation below shows the force exerted by the engine of the plane as it starts from rest from its initial position (θ = 0 radian) to the final position (θ = π radians)

$$F(\theta) = 50 + 2\theta$$

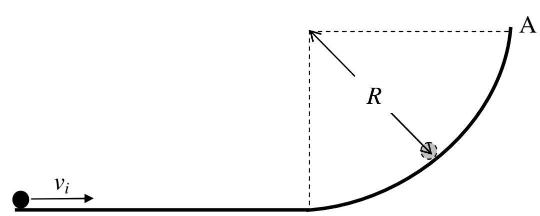
(F is in Newton and θ is in radian)

Find the work done by the engine when the plane travels from its initial point to the final point.

Figure 5.20 Exercise 4 of the standard material 4

Exercise 4 – Standard material 4

A hoop radius r = 1 cm and mass m = 2 kg is rolling at an initial speed v_i of 10 m/s along a track as shown. It hits a curved section (radius R = 2.0 m) and is launched vertically at point A.

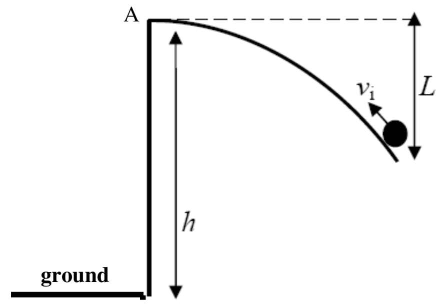


What is the launch speed of the hoop as it leaves the slope at point A?

Figure 5.21 Exercise 5 of the standard material 4

Exercise 5 – Standard material 4

A hoop of mass 0.5 kg starts with speed v_i= 12 m/s and rolls without slipping up a slope of height L = 6.0 m and is launched horizontally at point A The point of launch is at a height h = 12 m above the ground.



What is the launch speed of the hoop as it leaves the slope at point A?

5.3 Experimental design

In the spring 2010 semester, we conducted five focus group learning interview (FOGLI) sessions to test the effectiveness of our tutorials in comparison with standard instructional materials, i.e. sample problems and written solutions. The pretest-posttest control group experimental design was used. Twenty five students enrolling in the first-semester calculus-based physics course (Engineering Physics 1) volunteered to participate in our study. Each student was paid $75 for their participation in the study. These 25 students were randomly assigned into either a control or a treatment group. The number of students in each group varied with each session, ranging from 8 to 10 students in the control group and 12 to 14 students in the treatment group. Most of the students were freshmen or sophomores in engineering majors.

Each FOGLI session occurred within 10 days after the students had taken an exam in the course. The topics covered in the FOGLI were also the topics covered in the most recent exam. In each of the 90-minute FOGLI sessions, for the first 15-20 minutes students individually attempted a pre-test consisting of an algebraic problem and a graphical problem. In the next 40–50 minutes, students in the treatment group worked on our tutorials, while students in the control group worked on isomorphic textbook exercises covering the same physics concepts and principles, and employing the same representations. Students in both groups were encouraged to discuss with their partners while doing the exercises. After completing each of the exercises, the students in the control group were provided with a printed solution of the exercise they had just completed. Students would then read through the solution and compared with their own solution. Students might also ask the facilitator to clarify information in the written solution. On the other side, the students in the treatment group were required to check-in with the facilitator after they had completed an exercise in the tutorial. The facilitator then had a short conversation with the students to elicit their ideas on solving the exercise. If the students got the correct answer and had reasonable strategy for solving the exercise, the facilitator would ask students about what ideas they had learn from doing that exercise, and how those ideas might help them solve other similar exercises. If the students did not get the correct answer to the exercise or used a flawed strategy for solving the exercise, the facilitator would then ask students questions to help them recognize their errors. The facilitator, however, did not tell the students the correct answer or the strategy to solve the exercise. Once the students got to the correct answer for the exercise, the facilitator would also ask them to reflect on what they had learned in the exercise and how that might help them solve similar exercises. In the last 15-20 minutes, students individually attempted the post-test which differed from the pre-test only in numerical values provided in the problem statements.

Table 5.3 Comparison of the experimental procedures taken by the control and the treatment groups

<table>
<tr><th>Group</th><th>Treatment group</th><th>Control group</th></tr>
<tr><td>Similarities</td><td colspan="2">- Students worked on the pre-test and post-test problems individually.
- Students worked in small groups on the exercises in the exercise sets.
- Students were asked to notify the facilitator after they had completed each exercise in the set.</td></tr>
<tr><td>Differences</td><td>- Students worked on the tutorials.
- Short conversation with facilitator after each exercise.
- The facilitator elicited students' ideas and provided hints if needed, but did not tell the answer.</td><td>- Students worked on the standard material.
- Printed solution provided after each exercise.
- The facilitator clarified the solutions if needed, but did not tell the answer.</td></tr>
</table>

In this chapter, we will examine the effectiveness of the tutorials 3 and 4 in comparison with standard instructional materials. These tutorials were tested in the FOGLI sessions 3 and 4, respectively.

5.4 Data sources and analysis

Students' worksheets of the pre-test, post-test, and the tutorial were collected. Rubrics were created to grade the pre-test and post-test problems in each FOGLI session. Each problem was graded separately on the physics aspect and the representation aspect. The maximum score on the physics aspect was 10 points and on the representation aspect was 8 points.

The general rubrics for grading the physics aspect and the representation aspect (algebraic and graphical) of the test problems (pre-test/post-test) were presented in Table 5.4 to Table 5.6 below.

Table 5.4 The general rubric for grading the physics aspect of a pre-test/post-test problem

Points	0	1	2	3
Physics Approach	Student makes no progress toward a correct solution	Student uses a physics approach which is very complicated to get a correct solution (ex: using Newton's 2^{nd} law with changing force)	Student uses a mixture of an appropriate approach and inappropriate one (ex: using BOTH Newton's 2^{nd} law and conservation of energy in problem with changing force)	Student uses an appropriate physics approach (ex: a physics approach that may lead to a correct solution)
Physics Equation	Student doesn't have any equation	Student misses two or more quantities from the correct equation OR has two or more incorrect equations of physics quantities.	Student misses ONE quantity from the correct equation OR has ONE incorrect equation of physics quantity.	Student has correct equations of physics principle and quantities (ex: having all involved quantities with their correct signs)
Value of Physical Quantity	Student plugs in two or more in correct values (including sign errors) into physical quantities in an equation	One of the following cases: - Student plugs in ONE incorrect value (or sign error) into a physical quantity in an equation - Student only plugs in a few value from all of the given information	Student plugs in ALL correct values with correct signs into physical quantities in an equation (ex: using vertical distance as *h* when calculating gravitation potential energy *mgh*)	
Math Manipula-tion	Major errors in arithmetic (ex: errors in finding roots of an equation)	Correct arithmetic or minor errors in arithmetic (ex: confusing signs)		

Points	0	1	2	3
Units of Physical Quantity	Incorrect units of physical quantities (ex: using Newton as unit of work, adding quantities of different units)	Correct units of physical quantities (ex: using Joules as unit of work, adding quantities of the same units)		

Table 5.5 The general rubric for grading the physics aspect of an algebraic problem

Points	0	1	2	3	4
Interpretation of Function	Incorrect interpretation or use of the given equation (ex: interpreting F(x) as "F times x")	Correct interpretation and use of the given equation (ex: interpret F(x) as "F is a function of x")			
Mathematical Operator	Choosing incorrect mathematical operator to calculate physical quantities (integrate x(t) to find v(t), or v(t) to find a(t); differentiate F(x) to find work or "spring constant)				Choosing the correct mathematical operator to calculate physical quantities from equation given (ex: differentiate x(t) to find v(t), or v(t) to find a(t); integrate F(x) to find work)
Setting up Calculation	Setting up an incorrect calculation to calculate the desired quantity (ex: setting up the integral $\int F(\theta)d\theta$ to find work).	Setting up a correct calculation to calculate the desired quantity (ex: setting up the integral $\int F(\theta)Rd\theta$ to find work). An integral in the form of $\int F(x)$ is acceptable.			

Points	0	1	2	3	4
Mathematical Manipulation	Major errors in calculating derivative or integral (ex: confuse between differentiating and integrating)	Minor error or correct manipulation of derivative or integral.			
Unit of Quantity	Incorrect unit of quantity found from the calculation with the given function (ex: using Newton as unit of integral of F(x)dx)	Correct unit of quantity found from the calculation of the given function (ex: using Joule as unit of integral of F(x))			

Table 5.6 The general rubric for grading the representation aspect of a graphical problem

	0	1	2	3	4
Gather Information from Graph	ALL values of quantities read off from graph are incorrect	ALL values of quantities read off from graph are correct			
Mapping Graph to Physics	Incorrect mapping of graph quantity to physics quantity (ex: velocity as area under x(t) vs. t graph)				Correct mapping of graph quantity to physics quantity (ex: velocity as slope of x(t) vs. t graph, work as area under F(x) vs. x graph)
Setting up Calculation	Setting up incorrect equation of graph quantity (ex: slope = y/x, where y and x are coordinates of one point)	Setting up correct equation of graph quantity (ex: slope = $\Delta y/\Delta x$ = rise/run)			

	0	1	2	3	4
Manipulation of Graph Process	Incorrect plugging in of values or incorrect mathematical calculation of graph quantity (ex: incorrect calculation of slope, area under the graph)	Correct plugging in values and correct mathematical calculation of graph quantity (ex: correct calculation of slope, area under the graph)			
Unit of Graph Quantity	Incorrect unit of physical quantity found from graph (ex: using Newton as unit of the area under F(x) vs. x graph)	Correct unit of physical quantity found from graph (ex: using Joules as unit of area under F(x) vs. x graph)			

Due to the small number of participants in each group, the non-parametric Mann-Whitney test (Field, 2009) was employed to test the significance of the difference between the scores of the two groups on the pre-test and post-test. The null hypothesis was that the scores of the two groups were not statistically significantly different.

5.5 Results

In this section, we will present the pre-test and post-test problems, the scores of students in each group, and the results of the Mann-Whitney test in FOGLI sessions 3 and 4 where we tested our tutorials 3 and 4.

5.5.1 Tutorial 3 Results

There were 9 students in the control group and 12 students in the treatment group in FOGLI session 3. The control and the treatment groups met at different times. Nine students in the control group were divided into four groups (one group of 3 students and three groups of 2 students each). Twelve students in the treatment group were divided into 5 groups (two groups of 3 students each and three groups of 2 students each). Students in both control and treatment groups were given the freedom to choose their partners.

In the first 20 minutes of the FOGLI session, all students in both control and treatment groups worked independently on the pre-test which consisted of a graphical problem and an algebraic problem presented in Figure 5.22 and Figure 5.23 below, respectively. The physics aspect of these problems involved the application of the work-kinetic energy theorem for a point mass. The representation aspect of these problems involved calculating the work done by a force using the integral and the area under the curve of force versus linear displacement.

Figure 5.22 The graphical problem in the pre-test of FOGLI session 3

A 0.05 kg bullet is loaded into a gun compressing a spring which has spring constant k = 5000 N/m. The gun is tilted vertically downward and the bullet is fired into a drum 5.0 m deep, filled with a liquid.

The barrel of the gun is frictionless. The magnitude of the resistance force provided by the liquid changes with depth as shown in the graph below. The bullet comes to rest at the bottom of the drum.

What is the spring compression x?

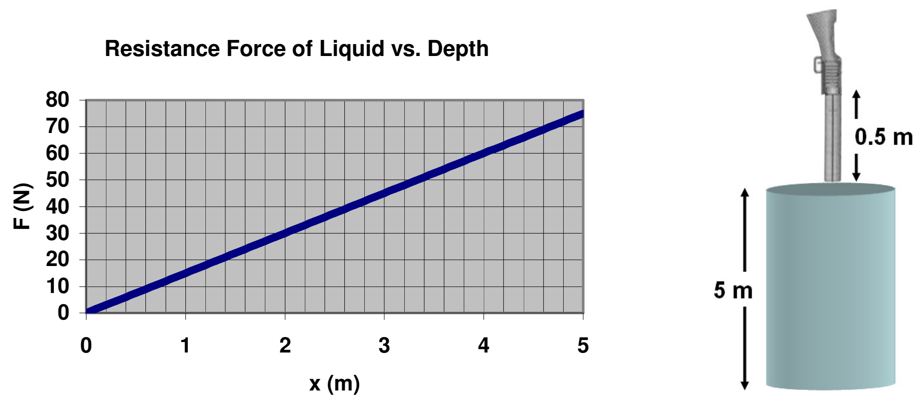


Figure 5.23 The algebraic problem in the pre-test of FOGLI session 3

A 0.05 kg bullet is loaded into a gun compressing a spring which has spring constant k = 5000 N/m. The gun is tilted vertically downward and the bullet is fired into a drum 5.0 m deep, filled with a liquid.

The barrel of the gun is frictionless. The magnitude of the resistance force *F* (in Newton) provided by the liquid changes with depth *x* (in meters) as per the following function:

$$F(x) = 8x + 0.5x^2$$

The bullet comes to rest at the bottom of the drum.

What is the spring compression *x*?

In the next 50 minutes, the students in the control groups worked on the set of exercises described in Figure 5.7 through Figure 5.11. Students were required to notify the facilitator after they completed each exercise. The facilitator then provided the students with the solution to the exercise they had just completed.

The students in the treatment group worked on the exercise set of our tutorial 3 (Figure 5.1 through Figure 5.6). Students were asked to check-in with the facilitator after they completed each exercise. The facilitator then engaged in a conversation with the students to elicit their ideas about the exercise and provided hints to help students solve the problem if needed, but did not tell them the solution.

All students in both the control and the treatment groups were able to solve the exercises easily. So the conversations between the facilitator and the students in the treatment group after the exercises were pretty short and the facilitator did not have to provide any hint to help students with the exercises of tutorial 3.

In the last 20 minutes of the FOGLI session, students in both the control and the treatment groups worked individually again on a graphical and an algebraic problems of the post-test, which were different from the pre-test problems only in numerical values of the quantities.

The rubrics for grading the physics and the representation aspects of the pre-test and post-test problems were built upon the general corresponding rubrics and are presented in the Table 5.7 through Table 5.9 below.

Table 5.7 Rubric for grading the physics aspect of the pre-test/post-test problems in FOGLI session 3

Points	0	1	2	3
Physics Approach	Student doesn't have any idea for solving the problem	Student uses Newton's 2nd law	Student uses a mixture of Conservation of energy AND Newton's 2nd law	Student uses conservation of energy OR Work-Kinetic Energy theorem
Physics Equation	Student has completely incorrect equation or doesn't have any equation	Student has ONE of the following: - Missing two or more quantities from the correct equation - Having two or more incorrect equations of physics quantities.	Student has ONE of the following: - Missing ONE quantity from the correct equation; - Having ONE incorrect equation of physics quantity. Example: one of the following equations: $\frac{1}{2}kx^2 = W_{nc}$; $\frac{1}{2}kx^2 + mgh = 0$; $kx + mgh = W_{nc}$ …	Student has the correct equation: $\frac{1}{2}kx^2 + mgh = W_{nc}$ OR its equivalence. Note: W_{nc} in this equation represents the "energy lost due to non-conservative forces" and therefore, W_{nc} has positive value.
Value of Physical Quantity	Student incorrectly loads two or more values with signs into physical quantities in an equation	Student incorrectly loads ONE value with sign into a physical quantity in an equation	Student correctly loads ALL values with correct signs into physical quantities in an equation. Note: h and W_{nc} may have positive or negative value depending on where student chose $h = 0$ and what he/she meant by W_{nc} (if W_{nc} is "energy lost to non-conservative forces", it gets positive value; if W_{nc} is "work by non-conservative forces", it gets negative value)	

Points	0	1	2	3
Mathematical Manipulation	Student makes major error or more than one minor error in arithmetic	Student makes correct arithmetic or ONE minor error in arithmetic - missing a square-root - missing a square - confusing signs		
Units of Physical Quantity	Student has incorrect unit of work, kinetic energy, potential energy, spring compression.	Student has correct unit of work, kinetic energy, potential energy, spring compression.		

Table 5.8 Rubric for grading the representation aspect of the algebraic pre-test/post-test problems in FOGLI session 3

Points	0	1	2	3	4
Interpretation of Function	One of the following cases: - Student doesn't indicate an understanding that F(x) is the equation of resistance force of the liquid with respect to depth. - Student plugs a specific value of x into F(x) and uses that as the force throughout the liquid (to calculate work W = F.d)	Student indicates an understanding that F(x) is equation of resistance force of liquid with respect to depth.			
Mathematical Operator	Student doesn't choose integration as a tool to find work OR divides F(x) by x to find the "spring constant" of the liquid so that they can find work by the liquid as ½ $k_{liquid}x^2$.	Student calculates integral of force but then uses it as the total force of the liquid on the bullet (multiplies by distance to find work)			Student calculates the integral of force equation over the depth of the liquid to find the work done by resistance force of liquid.

Points	0	1	2	3	4
Setting up Calculation	Student sets up an incorrect integral/limits OR incorrect calculation with the mathematical operator chosen in the previous step.	One of the following cases: - Student sets up the correct integral to find value of work done by the liquid: $\int_0^5 F(x)\,dx$. It is acceptable that student writes $\int_0^5 F(x)$. - Student sets up correct calculation with the mathematical operator chosen in the previous step.			
Mathematical Manipulation	Major errors in calculating the integral set up in the previous step (ex: confuse between differentiating and integrating)	Correct calculation of the integral set up in the previous step.			
Unit of Quantity	Incorrect unit of the quantity found from the calculation with the force equation.	Correct unit of the quantity found from the calculation with the force equation.			

Table 5.9 Rubric for grading the representation aspect of the graphical pre-test/post-test problems in FOGLI session 3

Points	0	1	2	3	4
Gather Information from Graph	More than one incorrect value from graph	Correct values of minimum and maximum force, depth read off from graph			
Mapping Graph to Physics	One of the following cases: - Student doesn't use the idea that area under the graph is work done by resistance force of liquid. - Student calculates the slope of graph and uses it as coefficient of resistance force of liquid (μ_k). - Student figures out the equation of the graph and uses it as coefficient of resistance force of liquid (μ_k).	Student calculates the area under the graph but uses it as total force by the liquid on the bullet (multiplies it with distance to find work)			One of the following cases: - Student calculates the area under the graph to find the work done by resistance force of liquid. - Student considers liquid as a spring whose "spring constant" is the slope of the graph. In this case, the area under the graph is the value of the term ½ $k_{liquid}x^2$ where x is the depth in the liquid. - Student calculates the integral of the equation of the graph to find the work done by resistance force of liquid.

Points	0	1	2	3	4
Setting up Calculation	One of the following: - Setting up incorrect area $base \times height$ - Setting up incorrect calculation of the slope of the graph - Setting up incorrect integral of graph's equation	One of the following: - Setting up correct area under the graph $\frac{1}{2} base \times height$ - Setting up correct calculation of the slope of the graph - Setting up correct integral of graph's equation			
Manipulation of Graph Process	One of the following: - Incorrect calculation of area set up in the previous step - Incorrect calculation of the slope set up in the previous step - Incorrect calculation of the integral set up in the previous step	One of the following: - Correct calculation of area set up in the previous step - Correct calculation of slope set up in the previous step - Correct calculation of the integral set up in the previous step			
Unit of Graph Quantity	Incorrect unit of area under graph or integral of graph's equation (Newton)	Correct unit of area under graph or integral of graph's equation (Joule)			

The inter-rater reliability of the rubric in Table 5.7 was 92%, the rubric in Table 5.8 was 95%, and the rubric in Table 5.9 was 85%. We present the results for the physics aspect and the representation aspect below.

5.5.1.1 The physics aspect

The means and standard deviations of the physics scores of each group in the pre-test and post-test are presented in Table 5.10. The Mann-Whitney test result is presented in Table 5.11.

Table 5.10 The mean (± SD) of the physics score of each group in the pre-test and post-test in FOGLI session 3

Problem	Group	Pre-test	Post-test
Graphical	Control	8.25 (± 2.25)	7.88 (± 2.80)
	Treatment	8.08 (± 2.78)	9.08 (± 1.31)
Algebraic	Control	8.13 (± 2.59)	8.50 (± 2.00)
	Treatment	8.33 (± 2.27)	9.17 (± 1.11)

Table 5.11 The Mann-Whitney test results for the physics score in the pre-test and post-test in FOGLI session 3

Problem	Pre-test	Post-test
Graphical	U = 48.5, p = 1.00, z = –0.04, r = –0.01	U = 59.0, p = 0.42, z = –0.85, r = –0.19
Algebraic	U = 51.5, p = 0.82, z = –0.27, r = –0.06	U = 57.0, p = 0.51, z = –0.69, r = –0.16

Table 5.11 indicates that there was no statistically significant difference in the physics scores between the control group and the treatment group in any problem of the pre-test and post-test. Although the effect sizes were slightly higher in the post-test (r = - 0.19 in the graphical problem and r = - 0.16 in the algebraic problem) than in the pre-test (r = - 0.01 and r = - 0.06 respectively), the effects were still weak. This implies that our tutorial 3 did not improve the students' ability to solve problems involving work – energy of a point mass significantly more than the control exercise set did. This result suggests that the tutorial 3 might need to be refined to increase students' practice with the work-energy theorem for a point mass.

5.5.1.2 Representation aspect

The means and standard deviations of the representation scores of each group in the pre-test and post-test are presented in Table 5.12. The Mann-Whitney test result is presented in Table 5.13.

Table 5.12 The mean (± SD) of the representation score of each group in the pre-test and post-test in FOGLI session 3

Problem	Group	Pre-test	Post-test
Graphical	Control	4.88 (± 2.75)	6.13 (± 1.89)
	Treatment	5.33 (± 2.84)	7.58 (± 0.90)
Algebraic	Control	4.25 (± 2.82)	4.88 (± 2.80)
	Treatment	4.08 (± 2.64)	7.00 (± 1.60)

Table 5.13 The Mann-Whitney test results for the representation score in the pre-test and post-test in FOGLI session 3

Problem	Pre-test	Post-test
Graphical	U = 52.0, p = 0.79, z = 0.31, r = -0.07	U = 74.5, *p = 0.04*, z = -2.04, r = -0.46
Algebraic	U = 46.5, p = 0.88, z = 0.11, r = 0.03	U = 73.5, *p = 0.05*, z = -1.97, r = -0.44

These tables indicate that the representation score of the treatment group was not statistically significantly different from that of the control group in the pre-test, but it was statistically significantly higher in the post-test ($p < .05$). The effect sizes, $r = -0.46$ in the graph problem and $r = -0.44$ in the equation problem in the post-test suggest that these were strong effects. This result implies that our tutorial 3 significantly improved students' ability to find work using the integral and the area under the curve of force versus linear displacement more than the control exercise set did.

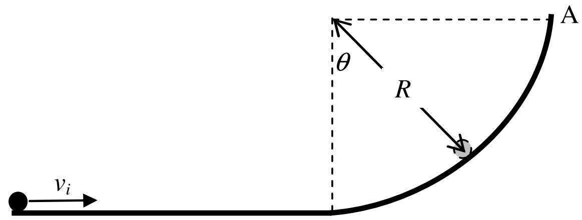
5.5.2 Tutorial 4 Results

There were 9 students in the control group and 13 students in the treatment group in FOGLI session 4. The control and the treatment groups met at different times. Nine students in the control group were divided into four groups (one group of 3 students and three groups of 2 students each). Thirteen students in the treatment group were divided into 6 groups (one group of 3 students each and five groups of 2 students each). Students in both the control and the treatment groups were given the freedom to choose their partners.

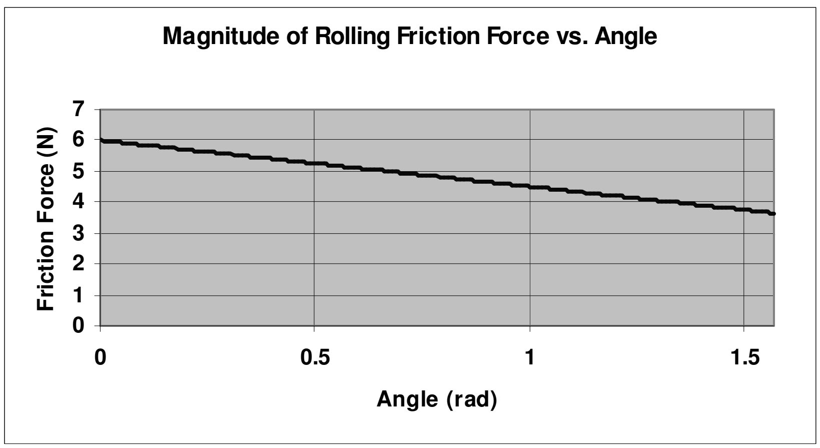
In the first 20 minutes of the FOGLI session 4, all students in both control and treatment groups worked independently on the pre-test which consisted of a graphical problem and an algebraic problem presented in Figure 5.24 and Figure 5.25 below, respectively. The physics aspect of these problems involved applying the work-kinetic energy theorem for a rigid body to calculate the linear speed of a sphere at the launch point. The representation aspect of these problems involved calculating the work done by the rolling friction force between the track and the sphere using the integral and the area under the curve.

Figure 5.24 The graphical problem in the pre-test of FOGLI session 4

A sphere radius r = 2 cm and mass m = 1.0 kg is rolling at an initial speed v_i = 10 m/s along a track as shown. It hits a curved section (radius R = 2.0 m) and is launched vertically at point A. The rolling friction on the straight section is negligible.



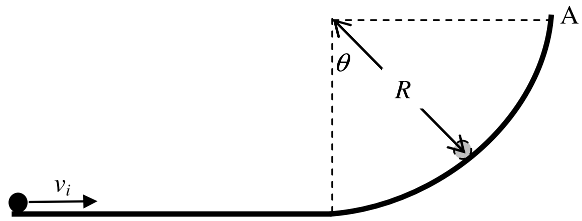
The magnitude of the rolling friction force acting on the sphere varies as angle θ as per the graph shown below



What is the launch speed of the sphere as it leaves the curve at point A?

Figure 5.25 The algebraic problem in the pre-test of FOGLI session 4

A sphere radius r = 2 cm and mass m = 1.0 kg is rolling at an initial speed v_i = 10 m/s along a track as shown. It hits a curved section (radius R = 2.0 m) and is launched vertically at point A. The rolling friction on the straight section is negligible.



The magnitude of the rolling friction force F (in Newton) acting on the sphere varies with angle θ (radians) as per the following function

$$F(\theta) = 5.0 - 1.5\theta$$

What is the launch speed of the sphere as it leaves the curve at point A?

In the next 50 minutes, the students in the control groups worked on the standard material 4 described in Figure 5.17 through Figure 5.21. Students were required to notify the facilitator after they completed each exercise. The facilitator then provided the students with the solution to the exercise they had just completed.

The students in the treatment group worked on the exercise set of our tutorial 4. Students were asked to check-in with the facilitator after they completed each exercise. The facilitator then engaged in a conversation with the students to elicit their ideas about the exercise and provided hints to help students solve the problem if needed, but did not tell them the solution.

Similar to FOGLI session 3, all students in both the control and the treatment groups were able to solve the exercises easily in FOGLI session 4. So the conversations between the facilitator and the students in the treatment group after the exercises were short and the facilitator did not have to provide any hint to help students with the exercises of tutorial 4.

In the last 20 minutes of the FOGLI session, students in both the control and the treatment groups worked individually again on a graphical and an algebraic problems of the post-test, which differed from the pre-test problems only in numerical values of the quantities.

The rubrics for grading the physics and the representation aspects of the pre-test and post-test problems in FOGLI session 4 were also built upon the corresponding general rubrics and are presented in the Table 5.14 through Table 5.16 below.

Table 5.14 Rubric for grading the physics aspect of the pre-test/post-test problems in FOGLI session 4

Points	0	1	2	3
Physics Approach	Student makes no progress toward a correct solution.	Student uses Newton's 2nd law	Student uses a mixture of Conservation of energy AND Newton's 2nd law	Student uses conservation of energy OR Work-Kinetic Energy theorem
Physics Equation	Student doesn't have any equation	Student has ONE of the following: - Missing TWO or more quantities from the correct equation - Having TWO or more incorrect equations of physics quantities.	Student has ONE of the following: - Missing ONE quantity from the correct equation; - Having ONE incorrect equation of physics quantity. Example: ONE of the following: - missing the rotational KE - missing the gravitational PE on the right-hand side - missing W_{nc} - using $\frac{2}{5}mr^2$ instead of $\frac{1}{2}I\omega^2$	Student has the correct equation: $\frac{1}{2}mv_i^2 + \frac{1}{2}I\omega_i^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2 + mgR + W_{nc}$ OR its equivalence. $(I_{sphere} = \frac{2}{5}mr^2)$ Note: W_{nc} in this equation represents the "energy lost due to non-conservative forces" and therefore, W_{nc} has positive value.

Value of Physical Quantity	Student plugs in TWO or more incorrect values (including sign errors) into physical quantities in an equation	Student plugs in ONE incorrect value (or makes a sign error) into a physical quantity in an equation	Student correctly plugs in ALL values with correct signs into physical quantities in an equation. Note: W_{nc} may have positive or negative value depending on what he/she meant by W_{nc} (if W_{nc} is "energy lost to non-conservative forces", it gets positive value; if W_{nc} is "work by non-conservative forces", it gets negative value)	
Mathematical Manipulation	Student makes major error or more than one minor error in arithmetic	Student makes correct arithmetic or ONE minor error in arithmetic - missing a square-root - missing a square - confusing signs		
Units of Physical Quantity	Student has incorrect unit of work, kinetic energy, potential energy.	Student has correct unit of work, kinetic energy, potential energy.		

Table 5.15 Rubric for grading the representation aspect of the algebraic pre-test/post-test problems in FOGLI session 4

Points	0	1	2	3	4
Interpretation of Function	Student doesn't indicate an understanding that F(θ) is equation of rolling friction force of the track with respect to angle OR plugs a specific value of θ into F(θ) and uses that as the force at every point on the track.	Student appropriately uses the fact that F(θ) is equation of rolling friction force of the track with respect to angle.			
Mathematical Operator	Student doesn't choose integration as the tool to find work OR uses the force equation to find the "coefficient of rolling friction" on the track.				Student calculates the integral of force equation over angle times radius of the track to find the work done by rolling friction force on the track.

Points	0	1	2	3	4
Setting up Calculation	One of the following: - Student sets up an incorrect integral (for example, $\int_0^{\pi/2} F(\theta)\,d\theta$ or $\int_0^{\pi/2} F(\theta)$ without multiplying the result by R afterward) - Student doesn't have an integral to find work done by the liquid.	Student sets up the correct integral to find value of work done by rolling friction: $\int_0^{\pi/2} F(\theta)R\,d\theta$. It is acceptable that student writes $\int_0^{\pi/2} F(\theta)R$. It is also acceptable that student writes the integrals $\int_0^{\pi/2} F(\theta)\,d\theta$ or $\int_0^{\pi/2} F(\theta)$ if later in the problem, student multiplies the value of that integral by the radius of the track.			
Mathematical Manipulation	Major errors in calculating the integral set up in the previous step (ex: confuse between differentiating and integrating)	Correct calculation of the integral set up in the previous step.			
Unit of Quantity	Incorrect unit of the quantity found from the integral of force equation (Newton)	Correct unit of the quantity found from the integral of force equation $\int_0^{\pi/2} F(\theta)R\,d\theta$ has unit of Joules $\int_0^{\pi/2} F(\theta)\,d\theta$ has unit of N.rad.			

Table 5.16 Rubric for grading the representation aspect of the graphical pre-test/post-test problem in FOGLI session 4

	0	1	2	3	4
Gather Information from Graph	Incorrect values of minimum and maximum forces (bases of trapezoid) and maximum angle (height of trapezoid)	Correct values of minimum and maximum forces (bases of trapezoid) and maximum angle (height of trapezoid)		Incorrect values of minimum and maximum forces (bases of trapezoid) and maximum angle (height of trapezoid)	Correct values of minimum and maximum forces (bases of trapezoid) and maximum angle (height of trapezoid)
Mapping Graph to Physics	Student doesn't use the idea that area under the graph or integral of the equation of the graph is part of the work done by rolling friction.	Student calculates the area under the graph or integral of the graph function but uses it as total force.	Student finds the area under the graph or integrate the equation of the graph and sets that equal work done by rolling friction (i.e. student doesn't multiply by the radius of the track)	Student doesn't use the idea that area under the graph or integral of the equation of the graph is part of the work done by rolling friction.	Student calculates the area under the graph or integral of the graph function but uses it as total force.

Setting up Calculation	One of the following cases: - Setting up incorrect area of trapezoid - Setting up incorrect integral of graph's equation	One of the following cases: - Setting up correct area under the graph $\frac{(base1+base2)}{2}\times height$ or area of triangle plus area of rectangle. - Setting up correct integral of graph's equation		One of the following cases: - Setting up incorrect area of trapezoid - Setting up incorrect integral of graph's equation	One of the following cases: - Setting up correct area under the graph $\frac{(base1+base2)}{2}\times height$ or area of triangle plus area of rectangle. - Setting up correct integral of graph's equation
Manipulation of Graph Process	One of the following cases: - Incorrect calculation of area set up in the previous step - Incorrect calculation of the integral set up in the previous step	One of the following cases: - Correct calculation of area set up in the previous step - Correct calculation of the integral set up in the previous step		One of the following cases: - Incorrect calculation of area set up in the previous step - Incorrect calculation of the integral set up in the previous step	One of the following cases: - Correct calculation of area set up in the previous step - Correct calculation of the integral set up in the previous step
Unit of Graph Quantity	Incorrect unit of area under graph or integral of graph's equation (Newton or Joule)	Correct unit of area under graph or integral of graph's equation (Newton time radian, or N.rad)		Incorrect unit of area under graph or integral of graph's equation (Newton or Joule)	Correct unit of area under graph or integral of graph's equation (Newton times radian, or N.rad)

The inter-rater reliability of the rubric in Table 5.14 was 88%, of the rubrics in Table 5.15 and Table 5.16 was 95% . We present the results for the physics aspect and the representation aspect below.

5.5.2.1 Physics aspect

The means and standard deviations of the physics scores of each group in the pre-test and post-test are presented in Table 5.17. The Mann-Whitney test result is presented in Table 5.18.

Table 5.17 The mean (± SD) of the physics score of each group in the pre-test and post-test in FOGLI session 4

Problem	Group	Pre-test	Post-test
Graphical	Control	4.89 (± 3.66)	7.00 (± 3.04)
	Treatment	6.54 (± 3.57)	8.77 (± 1.09)
Algebraic	Control	3.78 (± 3.31)	5.11 (± 4.31)
	Treatment	6.08 (± 3.95)	8.62 (± 1.39)

Table 5.18 The Mann-Whitney test results for the physics score in the pre-test and post-test in FOGLI session 4

Problem	Pre-test	Post-test
Graphical	U = 48.5, p = 1.00, z = –0.04, r = –0.01	U = 59.0, p = 0.42, z = –0.85, r = –0.19
Algebraic	U = 51.5, p = 0.82, z = –0.27, r = –0.06	U = 57.0, p = 0.51, z = –0.69, r = –0.16

These tables show similar trends of the physics score in FOGLI session 4 as in FOGLI session 3, so the conclusions are the same: our tutorial in this session didn't improve students' ability to solve problems involving the work-kinetic energy theorem for a rigid body in comparison to the control problem set. The treatment should be refined to increase students' practice with the work-kinetic energy theorem for a rigid body.

5.5.2.2 Representation aspect

The means and standard deviations of the representation scores of each group in the pre-test and post-test in FOGLI session 4 are presented in Table 5.19. The Mann-Whitney test result is presented in Table 5.20.

Table 5.19 The mean (± SD) of the representation score of each group in the pre-test and post-test in FOGLI session 4

Problem	Group	Pre-test	Post-test
Graphical	Control	2.00 (± 2.45)	3.78 (± 2.68)
	Treatment	3.08 (± 2.56)	5.92 (± 2.81)
Algebraic	Control	3.22 (± 2.22)	4.56 (± 2.24)
	Treatment	3.54 (± 1.45)	7.00 (± 1.53)

Table 5.20 The Mann-Whitney test results for the representation score in the pre-test and post-test in FOGLI session 4

Problem	Pre-test	Post-test
Graphical	U = 40.0, p = 0.20, z = -1.29, r = -0.28	U = 28.0, *p = 0.04*, z = -2.07, r = -0.44
Algebraic	U = 58.5, p = 1.00, z = -0.00, r = -0.00	U = 20.0, *p = 0.01*, z = -2.65, r = -0.56

These tables also indicate similar trend in the representation score as in FOGLI session 3, so the same conclusions apply: our tutorial in session 4 significantly improved students' ability to find work using the integral and then area under the curve of force versus angular displacement more than the control problem set did.

5.6 Conclusion

We created tutorials to facilitate students' learning to solve work – energy problems involving the integral and the area under the curve concepts. Each tutorial consisted of a set of exercises and a protocol for the conversation between the facilitator and the students after they had completed each of the exercises. An exercise set consisted of two or three pairs of matched math and physics exercises, a debate problem, and two problem posing tasks. The purpose of the pairs of matched math and physics exercises were to help students recall a mathematical model and then applied to a simple physics context. The debate problem was intended to prepare the students with the physics background necessary to solve typical work – energy problems, and to call for students' awareness on possible errors they might make when solving those problems. The problem posing tasks provided an opportunity for students to incorporate a mathematical model with a physics scenario to make a complete problem in which the mathematical model was employed. These tasks might provide students with a better view of how a mathematical model could be applied to a physics problem. All of the exercises in the tutorials were pretty simple so all groups were able to get the correct answers. Therefore the conversation between the facilitator and the students that took place after each exercise was mostly to elicit students' ideas about the exercise and what they had learned from the exercise. The facilitator did not have to provide hints to help students solve the exercises in the tutorial.

We conducted focus group learning interviews (FOGLI's) to test the effectiveness of our tutorials in comparison to standard instructional materials. The standard materials consisted of typical end-of-chapter exercises and solutions that covered the same concepts, principles, and had the same representations as the exercises in the tutorials. We found that both of our tutorials on the topics of work – energy for a point mass and for a rigid body significantly improved students' ability to calculate a physical quantity using the integral and the area under the curve concepts in a physics problem, although they were not so effective in preparing students with the physics background of the work – energy problems. These results suggested that the tutorials should be improved to better prepare students with the physics background of the problems.

5.7 Limitations and future work

The main limitation of this study is the small sample size of students with whom these tutorial materials were implemented. There were only 25 out of more than 200 students in the course participated in the study. Another limitation of the study was that there were only two tutorials, 90 minutes each, on the topic. These tutorials might create some improvement on students' performance on the tests as in our FOGLI sessions, but it is not likely that such short-term treatment could have a long-term effect on students' application of the integral in work-energy problems.

In future implementations, we plan to scale up the study to include a larger sample size. We plan to revise the tutorials, especially the physics aspect, and to implement them on a larger sample of students.

Chapter 6 - Tutorials to facilitate students' application of the integral concept to physics problems in electricity

6.1 Motivation and Introduction

In the fall 2009 study, we investigated students' difficulties in solving physics problems in electricity involving integration. During the individual interviews, students were asked to solve problems in which they had to set up and compute an integral to calculate a physical quantity. We found that the majority of the students were able to recognize the need for an integral in a problem. However, they had significant difficulties setting up and computing the desired integral. These difficulties occurred when the students attempted to set up the expression for the infinitesimal quantity and add up the infinitesimal quantities using the integral. These difficulties might be attributed primarily to students' inability to interpret the meaning of the infinitesimal term dx in the integral, and to students' disregard of how the infinitesimal quantities must be added up.

Many students in our interviews, however, were eventually able to solve the interview problems with verbal hints provided by the interviewer. This suggested that, with appropriate scaffolding, students would have been able to set up and compute the desired integral in electricity problems.

Based on the knowledge of the difficulties that students encountered and the scaffolding that were helpful, we proposed a strategy to facilitate students to apply the integral concept to physics problems in electricity. Specifically, our strategy aimed at helping students learn the meaning of the infinitesimal term dx in the integral and the accumulation process underlying the integral. Based upon this strategy, we developed instructional materials, which will be referred to as tutorials, to facilitate students' learning to apply the integral electricity problems. Each tutorial had two components:

- a problem segmented into a sequence of smaller, related exercises which led students through the procedure of solving the problem. As students solved the exercises in the sequence, they could learn the meaning of each term in the integral as well as the accumulative nature of the integral.

- a protocol for the conversation between the facilitator and the students to clarify the tutorial and students' ideas as they worked through the exercises.

In the fall 2010 semester, we created three tutorials on different topics of introductory electricity as follows:

- Tutorial 1: Electric field due to a charge distribution
- Tutorial 2: Resistance and capacitance
- Tutorial 3: Electric current

We tested the effectiveness of our tutorials in comparison with standard learning materials. The standard materials consisted of the same problems as in the tutorials but were not segmented, and the solutions to those problems.

In this chapter, we will present the rationale of our tutorials, and their impact on students' ability to apply the integral to physics problems in electricity. The research question for this study is: To what extent did our tutorials help students improve their ability to apply the integral concept to electricity problems, in comparison to standard materials (i.e. sample problems and solutions)?

6.2 Rationale of the tutorials and the standard materials

The central idea of the integral is the accumulation process, i.e. adding up an infinite number of individual amounts of a physical quantity to obtain the total amount of that quantity (e.g. resistance, current) or adding up an infinite number of individual effects to obtain the total effect (e.g. work, electric field). So the first crucial step is to set up the expression for the individual quantity or effect. Each of our tutorials aimed at helping students learn about integration via doing a sample problem involving integration. However, students were not asked to solve the problem as they usually were in the course. Instead, the problem was segmented into several exercises which led to the complete solution to the problem. Our tutorial starts with an exercise asking students to calculate the total value of a physical quantity (e.g. resistance, current) of some individual objects. The follow-up exercises were variations of the first exercise in which the individual objects evolve to become infinitesimal parts of a larger object. Solving these exercises, students might learn how the whole quantity of an object becomes an infinitesimal quantity of a larger object and how a sum becomes an integral.

Upon completion of each exercise, students were asked to check with a facilitator before proceeding to the next exercise. The facilitator then asked students to explain their solution to the exercise to make sure that students were on the right track. The facilitator might also provide students with verbal hints to help them recognize their errors if they did not get the correct result.

The standard material was intended to be similar to the material commonly used in the course for practice on a particular topic or skill. It contained the same problem as the one in the tutorial but the problem was not segmented. With this kind of material, students learned the topic or skill by attempting to solve the problem and then referring to its solution. The facilitator, in the role of the course instructor, was available to help the students clarify the problem and the solution, but did not explicitly teach the students how to solve the problem. Students learned to solve the problem by reading the solution which contained all necessary information, and reflected on their own solution.

In the next sub-section, we will discuss the rationale of each tutorial we created and the standard material we used for comparison with our tutorial.

6.2.1 Creation of the tutorial 1and the standard material 1

6.2.1.1 Creation of the tutorial 1

The structure of the tutorial 1 was different from the structure of the other two tutorials. It contained not only one problem segmented into several exercises, but also other mathematics and physics problems which aimed at emphasizing the accumulative nature of the integral. Problem 1 of this tutorial (Figure 6.1) was a simple math problem which asked students to calculate the area of the shaded region on the graph. This area could be found easily by counting the number of squares that made up the region and multiplying by the area each square represented. Problem 2 (Figure 6.2) also asked students to calculate the area of a shaded region, but the upper bound of this region was described by a continuous function. The area of this region, therefore, could not be obtained by just counting the squares, but by integrating the function of the upper bound. These two problems were intended to remind students of a basic knowledge in calculus: the area under a curve equaled the integral of the function of the curve. More importantly, these problems gave students an idea about how a sum of discrete elements became an integral when the elements became continuous. Problems 3 (Figure 6.3) and problem 4 (Figure 6.4) repeated this idea within a physics context. Problem 3 asked students to find the net electric field due to a series of point charges located at discrete positions on the x-axis at a point out of the x-axis. Because the charge distribution was discrete, the net electric field vector could be obtained by adding up the electric field vectors due to all individual charges. Due to the symmetry of the charge distribution, only the y-components of each electric field vectors contributed to the net field vector. Problem 4 was similar to problem 3 except that the charge distribution was continuous and was described by the charge distribution λ. So the procedure for solving problem 4 was similar to that for problem 3 except that the sum of the electric fields due to all point charges now became the integral of the electric field due to each infinitesimal charge. The pair of problems 3 and 4 emphasized the accumulative nature of the integral: the integral was the sum of an infinite number of quantities. Although the underpinnings of these two problems were the same, the problem 4 was more complicated because it required viewing the rod of charge as a continuous series of point charges and setting up an integral for the net electric field. This task was found to cause significant difficulties for the students in our interviews in the fall 2009 study, so the problem 4 was segmented into several steps. Each step asked students to complete a specific task which led to the solution to the problem.

Problem 5 (Figure 6.5) was another problem on finding the net electric field due to continuous charge distribution. This problem was also segmented into steps as problem 4. It provided students with another example of the process they had learned in problem 4.

Overall, the idea for the tutorial 1 was to lead students from the case of discrete quantity to the case of continuous quantity, during which a sum became an integral. The problems with the continuous cases were segmented into steps, which emphasized the contribution of each step to the solution to the problem.

Figure 6.1 Problem 1 of the tutorial 1

Problem 1 – Tutorial 1

Find the value of the shaded area below.

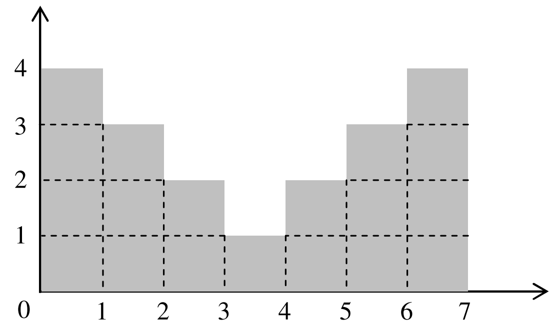


Figure 6.2 Problem 2 of the tutorial 1

Problem 2 – Tutorial 1

Find the value of the area below the curve of $f(x) = x^2 - 6x + 10$ from x = 0 to x = 6.

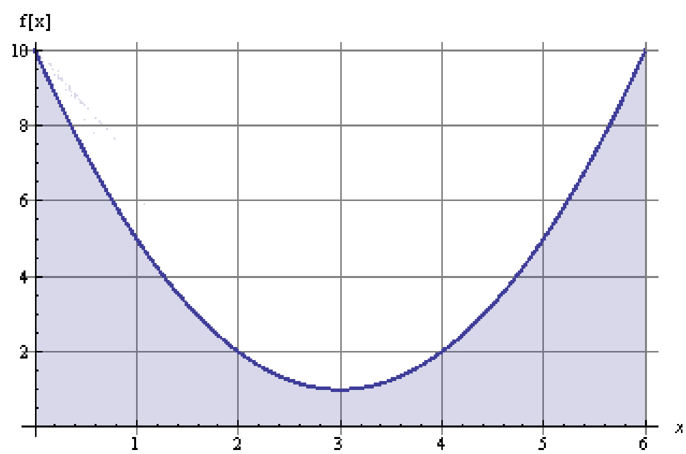


Figure 6.3 Problem 3 of the tutorial 1

Problem 3 – Tutorial 1

Five charges are placed along the x-axis at x = 1 m, 2 m, 3 m, 4 m, 5 m. The value of each charge is given as: $q = +3|x-3|$ where x is the location of the charge (x is in m, q is in μC). Find the electric field due to this system of charge at a point P which has x_P = 3.0 m, y_P = 3.0 m.

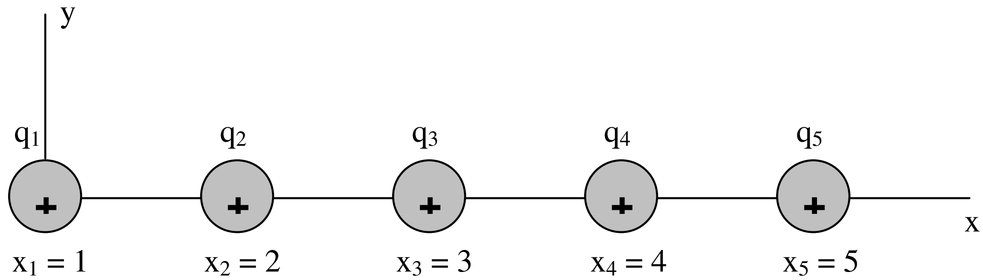
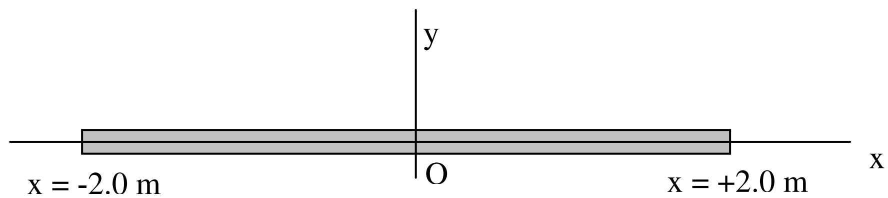


Figure 6.4 Problem 4 of the tutorial 1

Problem 4 – Tutorial 1

A non-conducting rod of length L = 4.0 m is lying along the x-axis from x = -2.0 m to x = +2.0 m and is having linear positive charge density λ. Find the electric field due to this rod at a point P which has $x_P = 0$, $y_P = +2.0$ m.



Please follow the steps below to solve this problem.

Step 1: Determine the sign and distribution of charge on the rod and locate point P

Step 2: Exploit symmetry to find the direction of the E-field

Step 3: Set up the expression for dE, the electric field due to a segment of charge dq at location x.

Step 4: Set up the expressions for dE_x, dE_y (the x- and y-components of dE)

Step 5: Set up the integral for finding the cumulative contribution of the component that adds up

Step 6: Express dq in terms of spatial variable (i.e. dx – the length segment along the rod) and the linear charge density $\lambda(x)$.

Step 7: Decide on the variable of integration and express all other variables in terms of the chosen variable

Step 8: Decide on the limits of integration and compute the integral

Step 9: Report the magnitude and direction of the E-field due to the rod at point P

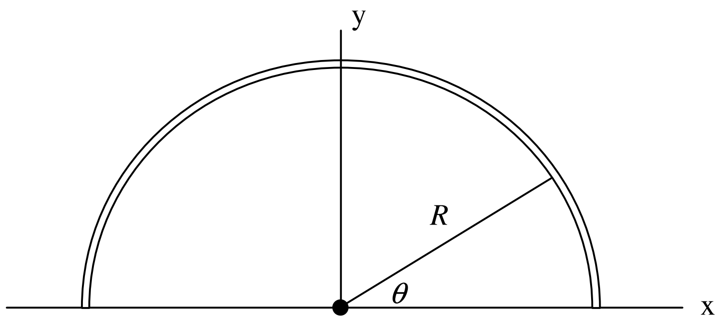
Figure 6.5 Problem 5 of the tutorial 1

Problem 5 – Tutorial 1

A non-conducting semi-circular arch carries charge with charge density given as per the function:

$$\lambda(\theta) = \lambda_0 \cos\theta$$

where λ_0 is a positive constant. Find the electric field due to this charge at the center of the arch.



Please follow the steps below to solve this problem.

Step 1: Determine the sign and distribution of charge on the arch

Step 2: Exploit symmetry to find the direction of the E-field

Step 3: Using Coulomb's Law, set up the expression for dE, the electric field due to a segment of charge dq at angle θ.

Step 4: Set up the expressions for dE_x, dE_y (the x- and y-components of dE)

Step 5: Set up the integral for finding the cumulative contribution of the component that adds up

Step 6: Express dq in terms of spatial variable (i.e. ds – the length segment along the arch spanning the angle dθ) and the linear charge density λ(θ)

Step 7: Decide on the variable of integration and express all other variables in terms of the chosen variable

Step 8: Decide on the limits of integration and compute the integral

Step 9: Report the magnitude and direction of the E-field due to the arch at its center

6.2.1.2 Creation of the standard material 1

The standard material 1 contained five problems on finding the total electric charge of two charge configurations (i.e. the rod and the arch) and the net electric fields due to those charge configurations. It differed from the tutorial 1 in several ways. First, it did not have any math problem. The math problems in the tutorial 1 were replaced by physics problems in algebraic and graphical representations. Second, the standard material 1 did not have problems with discrete distribution of charge. The problems with discrete distribution in the tutorial 1 were replaced by problems with continuous distribution. Third, the problems in the standard material were not segmented into steps.

All of the problems and their solutions in the standard material 1 are presented in Figure 6.6 through Figure 6.15. The first three problems (1, 2, and 3) asked students to find the total charge on a rod having a charge distribution described by a graph or an equation, and to find the net electric field due to that rod at a point out of it. The last two problems (4 and 5) in this set asked students to find the total charge on an arch having a charge distribution given as a function and the net electric field due to that arch at its center. The problems on the electric field (i.e. problems 3 and 5) were adopted from the tutorial 1, while the other three problems on the total charge were added to the standard material to replace the math problems and the problems with discrete distribution in the tutorial. Note that although all of the problems in the standard material dealt with continuous distribution, none of them were segmented. These problems were similar to end-of-chapter practice problems in a typical textbook.

Figure 6.6 Problem 1 of the standard material 1

Problem 1 – Standard material 1

Find the total charge on a non-conducting rod of length L = 4.0 m lying along the x-axis and having linear charge density given as per the graph below.

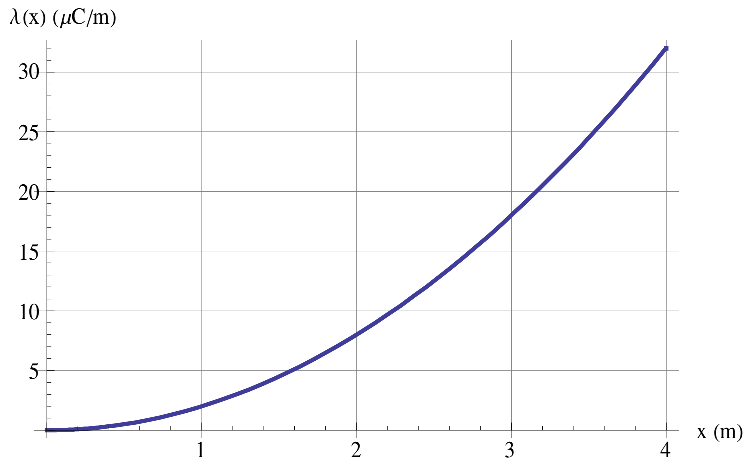
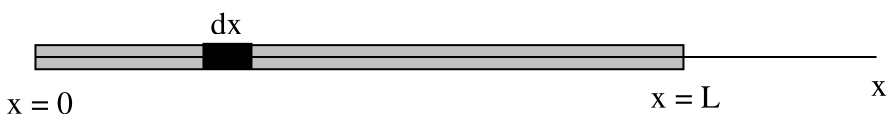


Figure 6.7 Solution to problem 1 of the standard material 1

Solution to Problem 1 – Standard material 1



Consider a small segment dx. This segment carries a charge: $dq = \lambda(x)\,dx$.

The total charge is found by integrating this dq over the length of the rod: $Q = \int dq = \int_0^4 \lambda(x)\,dx$.

This integral is equal to the area under the curve of λ(x) vs. x, which is the curve in the given graph. This area can be approximated by counting the rectangles. There are approximately 8.5 rectangles under the curve; each rectangle represents a quantity of:

$$(1.0\,m).(5.0\,\mu C/m) = 5.0\,\mu C$$

So the total area is 8.5x(5.0 μC) = 42.5 μC.
So the total charge on the arch is: Q = 42.5 μC.

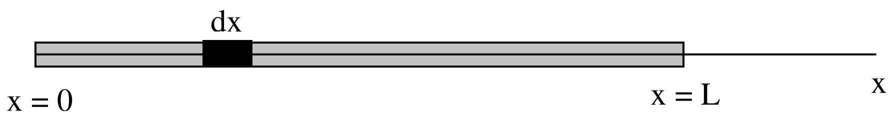
Figure 6.8 Problem 2 of the standard material 1

Problem 2 – Standard material 1

Find the total charge on a non-conducting rod of length L = 4.0 m lying along the x-axis and having linear charge density $\lambda(x) = 2x^3$ (x is in m, λ is in μC/m).

Figure 6.9 Solution to problem 2 of the standard material 1

Solution to Problem 2 – Standard material 1



Consider a small segment dx. This segment carries a charge: $dq = \lambda(x)\,dx$.

The total charge is found by integrating this dq over the length of the rod:

$$Q = \int dq = \int_0^4 \lambda(x)\,dx$$

$$Q = \int_0^4 2x^3 dx = \left[2\frac{x^4}{4}\right]_0^4 = 128\,\mu C$$

So the total charge on the arch is: Q = 128 μC.

Figure 6.10 Problem 3 of the standard material 1

Problem 3 – Standard material 1

A non-conducting rod of length L = 4.0 m is lying along the x-axis from x = -2.0 m to x = +2.0 m and is having linear positive charge density λ. Find the electric field due to this rod at a point P which has $x_P = 0$, $y_P = +2.0$ m.

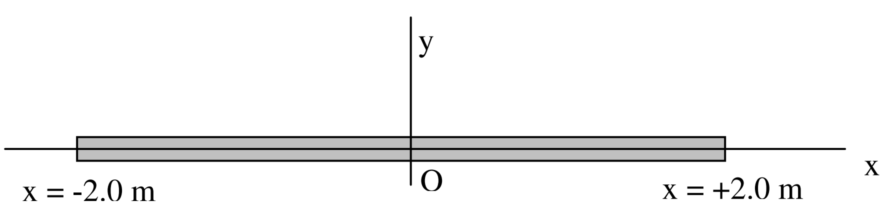
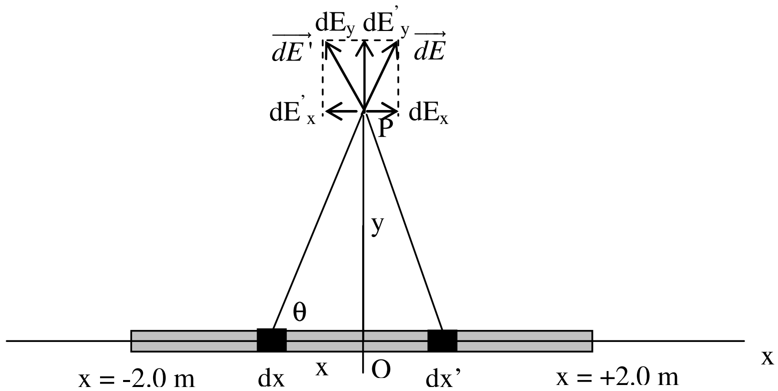


Figure 6.11 Solution to the problem 3 of the standard material

Solution to Problem 3 – Standard material 1



Consider a small segment dx along the rod at location x. This segment carries a charge: $dq = \lambda dx$.

This charge causes at P an electric filed $\overrightarrow{dE}$ which has the magnitude: $dE = k\dfrac{dq}{r^2} = k\dfrac{\lambda dx}{r^2}$.

$\overrightarrow{dE}$ can be broken into two components: $dE_x = dE.\cos\theta$ and $dE_y = dE.\sin\theta$

With any charge dq on the right half of the rod, there is a charge dq' on the left half of the rod which equals to dq. The charge dq' causing at O an electric field $\overrightarrow{dE'}$ which can be broken into dE'_x and dE'_y. The x-components of $\overrightarrow{dE}$ and $\overrightarrow{dE'}$ cancel out, while their y-components add up. So the total electric field due to the arch points in the +y direction and has the magnitude which is the integral of dE_y over the rod.

$$E_y = \int dE_y = \int dE\sin\theta = \int k\frac{\lambda dx}{x^2+y^2}\sin\theta = k\alpha\int\frac{\lambda}{x^2+y^2}\sin\theta dx$$

We will integrate with respect to x, so we'll write sinθ in terms of x, which is: $\sin\theta = \dfrac{y}{x} = \dfrac{y}{\left(x^2+y^2\right)^{1/2}}$

So the integral becomes: $E_y = k\displaystyle\int\frac{\lambda}{x^2+y^2}\frac{y}{\left(x^2+y^2\right)^{1/2}}dx = ky\lambda\int_{-2}^{2}\frac{dx}{\left(x^2+y^2\right)^{3/2}}$, (y = 2.0 m)

$$E_y = ky\lambda\left[\frac{x}{y^2\sqrt{x^2+y^2}}\right]_{-2}^{2} = \frac{k\lambda}{2}\left[\frac{2}{\sqrt{8}} - \frac{-2}{\sqrt{8}}\right] = \frac{k\lambda}{\sqrt{2}}$$

So, the electric field due to the arch at its center O is $\overrightarrow{E_O} = +\dfrac{k\lambda}{\sqrt{2}}\hat{j}$.

Figure 6.12 Problem 4 of the standard material 1

Problem 4 – Standard material 1

A non-conducting semi-circular arch carries charge with charge density given as per the function:

$$\lambda(\theta) = \lambda_0 \cos\theta$$

where λ_0 is a positive constant. Find the total charge on the arch.

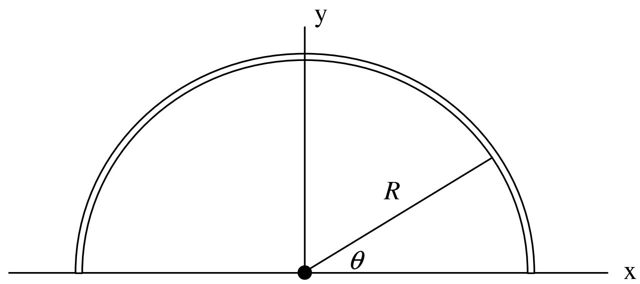
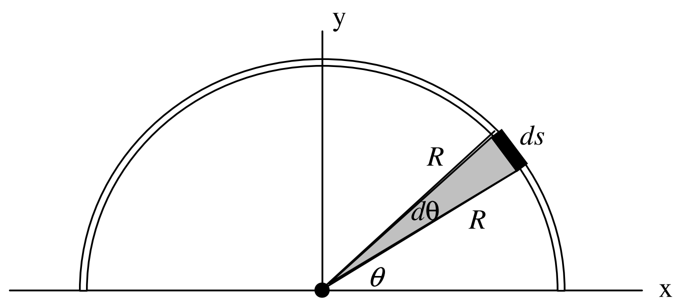


Figure 6.13 Solution to problem 4 of the standard material 1

Solution to Problem 4 – Standard material 1



Consider a small segment ds along the arch spanning an angle dθ. We have the relation: $ds = Rd\theta$.

This segment carries a charge: $dq = \lambda(\theta)ds = \lambda(\theta)Rd\theta$.

The total charge is found by integrating this dq over all angle of the arch:

$$Q = \int dq = \int_0^\pi \lambda(\theta)Rd\theta,\ Q = R\int_0^\pi \lambda(\theta)d\theta = R\int_0^\pi \lambda_0 \cos\theta d\theta = \lambda_0 R\int_0^\pi \cos\theta d\theta$$

$$Q = \lambda_0 R\int_0^\pi \cos\theta d\theta = \lambda_0 R[\sin\theta]_0^\pi = 0$$

The total charge on the arch is zero.

This result makes sense because on the right half of the arch where cosθ is positive, the charge is positive, and on the left half of the arch where cosθ is negative, the charge is negative. The cosine function is symmetric with respect to θ = π/2, so the positive charge on the right half is equal in magnitude to the negative charge on the left half, which results in zero net charge.

Figure 6.14 Problem 5 of the standard material 1

Problem 5 – Standard material 1

A non-conducting semi-circular arch carries charge with charge density given as per the function:

$$\lambda(\theta) = \lambda_0 \cos\theta$$

where λ_0 is a positive constant. Find the electric field due to this charge at the center of the arch.

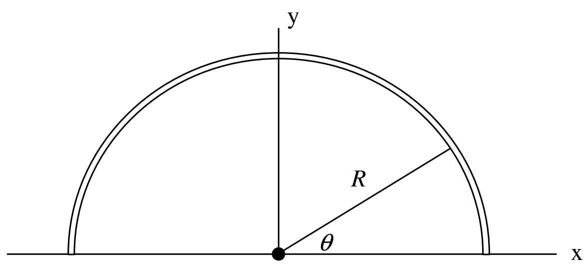
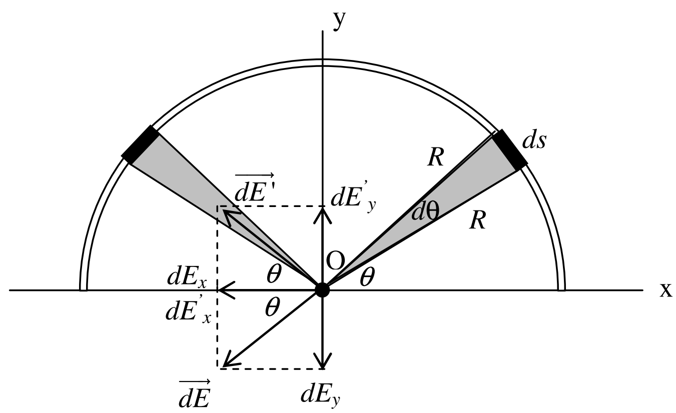


Figure 6.15 Solution to problem 5 of the standard material 1

Solution to Problem 5 – Standard material 1



The cosine function is positive from θ = 0 to θ = π/2 and is negative from θ = π/2 to θ = pi, so the charge on the right half of the arch is positive while the charge on the left hafl of the arch is negative.

Consider a small segment ds along the arch spanning an angle dθ. We have the relation: $ds = Rd\theta$.

This segment carries a charge: $dq = \lambda(\theta)ds = \lambda(\theta)Rd\theta$.

This charge causes at O an electric filed $\overrightarrow{dE}$ which has the magnitude:

$$dE = k\frac{dq}{R^2} = k\frac{\lambda(\theta)Rd\theta}{R^2} = k\frac{\lambda(\theta)d\theta}{R}.$$

$\overrightarrow{dE}$ can be broken into two components: $dE_x = dE.\cos\theta$ and $dE_y = dE.\sin\theta$

The cosine function is symmetric about θ = π/2, so the charge is distributed symmetrically about the top of the arch. This means that with any charge dq on the right half of the arch, there is a charge dq' on the left half of the arch which equals in magnitude with dq. The charge dq' causing at O an electric field $\overrightarrow{dE'}$ which can be broken into dE'_x and dE'_y. The y-components of $\overrightarrow{dE}$ and $\overrightarrow{dE'}$ cancel out, while their x-components add up. So the total electric field due to the arch points in the –x direction and has the magnitude which is the integral of dE_x over all angle on the arch.

$$E_x = \int dE_x = \int dE\cos\theta = \int k\frac{\lambda(\theta)d\theta}{R}\cos\theta = \int k\frac{\lambda_0\cos\theta d\theta}{R}\cos\theta$$

$$E_x = \frac{k\lambda_0}{R}\int_0^\pi \cos^2\theta d\theta = \frac{k\lambda_0}{R}\int_0^\pi \frac{1+\cos 2\theta}{2}d\theta = \frac{k\lambda_0}{2R}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^\pi$$

$$= \frac{k\lambda_0}{2R}\left[\left(\pi + \frac{1}{2}\sin 2\pi\right) - \left(0 + \frac{1}{2}\sin 0\right)\right] = \frac{k\lambda_0\pi}{2R}$$

So, the electric field due to the arch at its center O is $\overrightarrow{E_O} = -\frac{k\lambda_0\pi}{2R}\hat{i}$

6.2.2 Creation of the tutorial 2 and the standard material 2

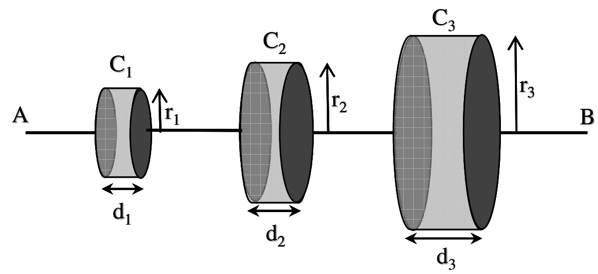
6.2.2.1 Creation of the tutorial 2

The tutorial 2 consisted of a sequence of exercises which were designed to help students learn about the integral in physics problem. The context of this tutorial was the problem of finding the capacitance of a capacitor whose plates were considerably far apart compared to the size of the plates. However, students were not asked to solve that problem. Instead, they were led through a sequence of exercises which asked them to find the equivalent capacitance of series of individual capacitors. These capacitors evolved from separate capacitors with different plate sizes and separations between the plates to adjacent capacitors with similar plate sizes and small separations. The result of the last exercise in the sequence was the capacitance of the capacitor with different plate sizes and large separation, which was the answer to the initial problem. The sequence of exercises in the tutorial 2 is presented in Figure 6.16 below.

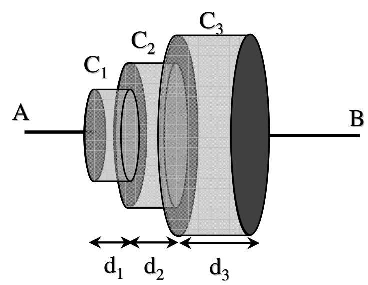
Figure 6.16 Sequence of exercises in the tutorial 2

Tutorial 2

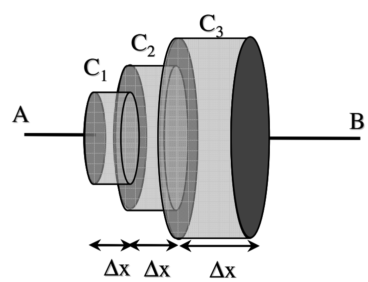
1. Three capacitors made of the same material (permittivity ε) have different distances, d_1, d_2, d_3 between their plates respectively. The plates of each capacitor are circular and have radii r_1, r_2, r_3 respectively. Find the total capacitance C_{AB}.



2. What is the total capacitance C_{AB}, if all three capacitors above are arranged side by side as shown in the figure below? [Hint: Is the result same as above?]



3. What is the total capacitance C_{AB} if all three capacitors above have the same distance between their plates, Δx?



4. What is the total capacitance C_{AB} if instead of three capacitors, there are a large number of capacitors of the same length Δx and the radius of the *i*-th capacitor plate located at x_i is $r_i = a + bx_i$?
5. What is the total capacitance C_{AB} if instead of three capacitors, there are a very large number of infinitesimally thin capacitors of the same distance between plates dx and the radius of plates of these capacitors varies as per the function $r(x) = a + bx$, where x is the location of the thin capacitor with respect to the left end of the wire and $0 \leq x \leq L$?

The tutorial started with a general case: finding the equivalent capacitance of a series of 3 capacitors having different plate sizes and different separations between the plates (Exercise 1). This exercise was to remind students of the formula for the capacitance of parallel-plate capacitor and the formula for the equivalent capacitance of a series of capacitors. Exercise 2 asked for the total capacitance of the same capacitors in exercise 1 when they were arranged side by side. Putting the capacitors together in exercise 2 triggered the idea that these capacitors might be a part of a larger system. In exercise 3, the separation between the plates of all capacitors was set to be equal to Δx. This change in the system introduced the common term among the capacitors: Δx, which would become the infinitesimal term dx when the separation became infinitesimally small. Exercise 4 generalized the case of exercise 3 for N capacitors with the same separation between the plates Δx but the plate sizes depended upon the position of the capacitor in the series as per an equation. Exercise 5 generalized the case of exercise 4, when the number of capacitor was infinite and the separation between the plates was infinitesimally thin. This last exercise was to help students learn that Δx became dx and the sum became the integral when the number of capacitor became infinite and the separation between their plates became infinitesimally small.

The reason for choosing the capacitance of series capacitors as the context for this tutorial was that the equivalent capacitance of a series of capacitors was found by adding the inverse of the individual capacitance, i.e. $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$. So when each capacitor became an infinitesimal element of a larger capacitor, the equivalent capacitance be calculated by integrating the inverse of the infinitesimal capacitance, i.e. $\frac{1}{C_{eq}} = \int \frac{1}{dC}$, instead of integrating $\int dC$ which was a common error that students made in our interviews in the fall 2009 study. So by working on our tutorial 2, students might learn that they must attend to how the infinitesimal quantities must be added up when doing an integral.

In summary, our tutorial 2 aimed at helping students learn the physical meaning of the infinitesimal term dx in the integral (i.e. the quantity it represented), the nature of the integral as an accumulation process, and the method for accumulating the infinitesimal quantities. The tutorial 2, therefore, targeted the most significant difficulties students expressed in our fall 2009 study.

6.2.2.1 Creation of the standard material 2

The standard material 2 consisted of the problem on the capacitance of a capacitor whose plates are considerably far apart compared to their sizes and its solution (Figure 6.17 and Figure 6.18). With this set of material, students learned by attempting to solve the capacitor problem on their own, then referred to the solution and reflected on their own solution.

Figure 6.17 Problem in the standard material 2

Standard material 2

Consider a capacitor of material, permittivity ε. The capacitor consists of two circular plates of radii *a* and *b* placed at a distance *L* apart.
Derive an expression for the capacitance of this capacitor in terms of its length *L*, radius *a*, radius *b*, and permittivity ε.

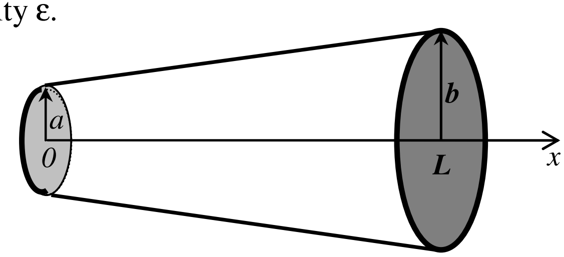
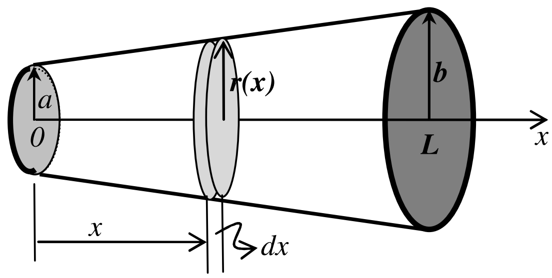


Figure 6.18 Solution to the problem in the standard material 2

Solution – Standard material 2

Imagine that there are several fictitious plates which are a distance *dx* apart in the region between the two plates of the capacitor. The given capacitor is now a *series* combination of several capacitors made by the fictitious plates.



If *dx* is small enough, then the radii of two adjacent plates are almost equal, so the capacitance of the capacitor made by these two adjacent plates is:

$$dC = \varepsilon_0 \frac{A}{dx} = \varepsilon_0 \frac{\pi\left[r(x)\right]^2}{dx}$$

Where:

$r(x)$ is the radius of the two plates of the fictitious capacitor located at *x*.

At $x = 0$, $r = a$, and at $x = L$, $r = b$, so the expression of $r(x)$ is: $r(x) = a + \frac{b-a}{L}x$

Then: $dC = \varepsilon_0 \frac{\pi\left[a + \frac{b-a}{L}x\right]^2}{dx}$

The given capacitor is a series combination of several fictitious capacitors, so its capacitance is the equivalent capacitance of all fictitious capacitors from $x = 0$ to $x = L$.

$$\frac{1}{C} = \int \frac{1}{dC} = \int_0^L \frac{dx}{\varepsilon_0 \pi\left[a + \frac{b-a}{L}x\right]^2} = \frac{1}{\varepsilon_0 \pi} \frac{L}{b-a}\left(-\frac{1}{a + \frac{b-a}{L}x}\right)\Bigg|_0^L$$

$$\frac{1}{C} = \frac{L}{\varepsilon_0 \pi (b-a)}\left(-\frac{1}{b} + \frac{1}{a}\right) = \frac{L}{\varepsilon_0 \pi (b-a)}\left(\frac{b-a}{ab}\right) = \frac{L}{\varepsilon_0 \pi ab}$$

So the capacitance of the given capacitor is: $\boxed{C = \frac{\varepsilon_0 \pi ab}{L}}$

6.2.3 Creation of the tutorial 3 and standard material 3

6.2.3.1 Creation of the tutorial 3

The tutorial 3 consisted of two physics problems. The topic of the tutorial 3 was finding the total current I in a wire from the current density j. The current in a wire could be calculated by the equations $I = jA$ if $j = const$, and $I = \int jdA$ if $j \neq const$, in which A was the cross-sectional area of the wire through which the current flew. We found from our fall 2009 interviews that the concept of current density was not easy to understand for many students, so they had difficulties interpreting the equations mentioned above. Students in our fall 2009 interviews were not familiar with calculating the cross-sectional area of some shapes, e.g. the cylindrical shell, so many of them needed help calculating the cross-sectional area A of the wire.

The first problem in the tutorial 3 aimed at targeting these difficulties. In part a of this problem, students were asked to calculate the cross-sectional area of a cylindrical shell having the inner radius R and thickness ΔR. Part b of this problem asked students to calculate the total current in the cylindrical shell given the constant current density $j = j_0$. With these two parts, the problem 1 in the tutorial 3 was intended to familiarize students with the concept of current density, the area of a cylindrical shell, and the method of finding the total current from the current density and the cross-sectional area.

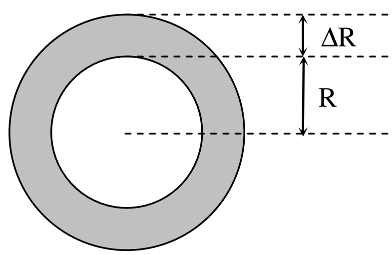
The problem 2 of the tutorial 3 was similar to the tutorial 2. It consisted of a sequence of exercises that led students from finding the current in an individual cylindrical shell to finding the current in the circular wire by adding up the current in each infinitesimal shell. The first exercise in the sequence asked students to calculate the total current in a wire that was made of two separate, coaxial cylindrical shells. The shells had inner radii R_1, R_2, same thickness ΔR, and carried currents with different current densities j_1, j_2 but in the same direction. In the next exercise, the radius of the smaller shell was increased such that there was no gap between the shells but the shells were insulated from each other so they still had different current densities. The last exercise generalized the case in exercise 2 to several infinitesimal shells having inner radius ranging from 0 to R and the thickness ΔR of each shell was very small that the current density in each shell could be considered constant. Working through this sequence of exercises, students were introduced the idea of modeling the cylindrical wire or shell as a combination of several infinitesimally thin shells on which the current density could be considered constant, so the total current in the wire or the shell was the sum of the current in each of these shells, which became an integral when the shells became infinitesimally thin.

In summary, our tutorial 3 aimed at helping students learn the physical meaning of the infinitesimal term dx in the integral (i.e. the quantity it represented), the nature of the integral as an accumulation process, and the method of accumulating the infinitesimal quantities. The tutorial 2, therefore, targeted the most significant difficulties students expressed in our fall 2009 study.

Figure 6.19 Problem 1 of the tutorial 3

Problem 1 – Tutorial 3

a. What is the cross-sectional area of a cylindrical shell which has inner radius R and thickness ΔR?



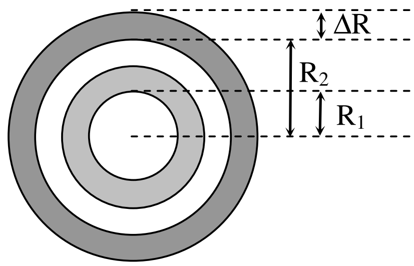
b. This shell carries a current which is distributed uniformly over the cross section of the shell with current density $j = j_0$. What is the total current in the shell?

Figure 6.20 Problem 2 of the tutorial 3

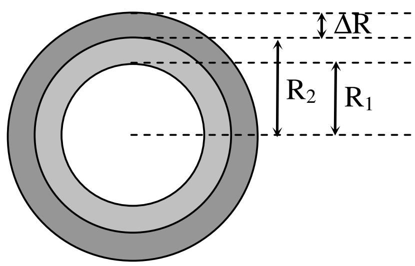
Problem 2 – Tutorial 3

A wire is made of two coaxial cylindrical conducting shells which have inner radii R_1, R_2 and the same thickness ΔR. These shells are carrying two currents flowing in the same direction with current densities j_1 and j_2 respectively (j_1, j_2 are constants).

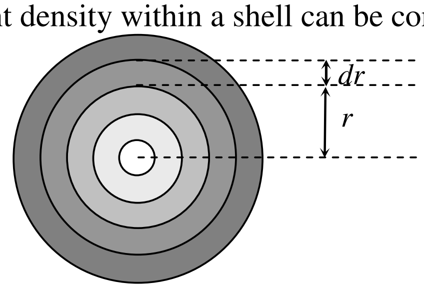
a. What is the total current in these two wires?



b. What is the total current in the wire if there is no gap between the shells? (the shells are insulated from each other)



c. What is the total current in the wire if the wire is made of several infinitesimally thin shells, each has inner radius *r*, thickness *dr* and current density $j(r)$. (Note that *dr* is small enough that the current density within a shell can be considered constant across)



6.2.3.1 Creation of the standard material 3

The standard material 3 consisted of two problems on finding the currents from the current density and their solutions. These problems were similar to the last exercises in each of the two sequences of exercise in the tutorial 3 and were typical textbook problems on the topic.

All of the problems and their solutions in the standard material 3 are presented in Figure 6.21 through Figure 6.24 below.

Figure 6.21 Problem 1 of the standard material 3

Problem 1 – Standard material 3

A long, cylindrical conducting shell has inner radius R_1 and outer radius R_2. It carries a current which is distributed uniformly across the cross-sectional area of the shell with current density $j = j_0$ going into the page. What is the total current in the shell?

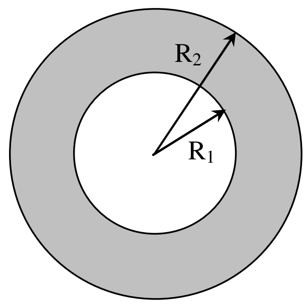


Figure 6.22 Solution to problem 1 of the standard material 3

Solution to Problem 1 – Standard material 3

The cross-sectional area of the shell is: $A = \pi R_2^2 - \pi R_1^2 = \pi\left(R_2^2 - R_1^2\right)$

The total current in the shell is then: $I_{total} = j.A = j_0\pi\left(R_2^2 - R_1^2\right)$

Figure 6.23 Problem 2 of the standard material 3

Problem 2 – Standard material 3

A long, cylindrical conducting shell has inner radius R_1 and outer radius R_2. It carries a current which has current density $j = \alpha r$ going into the page. What is the total current in the shell.

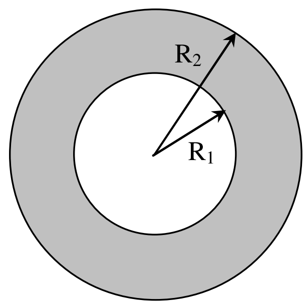
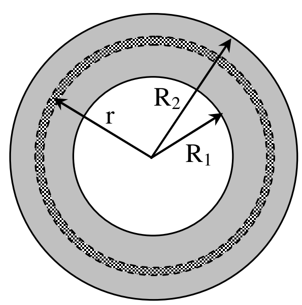


Figure 6.24 Solution to problem 2 of the standard material 3

Solution to Problem 2 – Standard material 3

Consider an infinitesimally thin ring which has inner radius *r* and thickness *dr* small enough that the current density *j(r)* is almost constant across the cross section of this ring.



The cross sectional area of this ring is:

$dA = \pi\left(r+dr\right)^2 - \pi r^2 = \pi\left(r^2 + 2r.dr + dr^2\right) - \pi r^2 = 2\pi r dr$ (note that *dr* is very small so $dr^2 \approx 0$)

The current through this cross sectional area is: $dI = j\left(r\right)dA = \alpha r 2\pi r dr = \alpha 2\pi r^2 dr$

The total current in the shell is then the integral of this infinitesimal current:

$$I_{total} = \int dI = \int_{R_1}^{R_2} \alpha 2\pi r^2 dr = \alpha 2\pi \left.\frac{r^3}{3}\right|_{R_1}^{R_2} = \alpha 2\pi\left(\frac{R_2^3 - R_1^3}{3}\right) = \frac{2}{3}\alpha\pi\left(R_2^3 - R_1^3\right)$$

6.3 Experimental design

In the fall 2010 semester, we tested the effectiveness of our tutorials in comparison with the standard materials on students enrolling in the Engineering Physics 2 course, which covered electricity and magnetism. An important portion of this course was the studio session, which was an integrated session of problem solving and hands-on experiments. All students in the course met for two 2-hour studio sessions per week. In each studio session, the first 40 minutes was for problem solving and the rest of the time for hands-on experiment.

The focus group learning interviews (FOGLI's) in the spring 2010 study were conducted as an independent activity outside of the course. That limited the number of students participating in our study, because the students had to meet outside of the course. So in order to involve all students in the course in our study in the fall 2010, we decided to conduct our experiments during the problem solving sessions in the studio. Students worked in group of 3 or 4 during the studio sessions, so it was similar to conducting a FOGLI, but with 10 groups instead of a few groups at a time. However, conducting the FOGLI during the studio sessions also put a constraint on the length of the FOGLI and hence the length of the treatments, because we had to ensure adequate time for regular activities of the studio sessions. For this reason, each FOGLI in the fall 2010 study lasted for only 40 - 50 minutes, instead of 90 minutes as in the spring 2010 FOGLI.

We also employed the pretest-posttest control group experimental design as in the spring 2010 study. There were a total of 220 students divided into 6 studio sections, so 3 sections (approximately 110 students) served as the control group, and the other 3 sections (approximately 110 students) served as the treatment group. In each of the 50-minute FOGLI's, for the first 10 minutes, students worked individually on a pre-test which was a problem involving integration on the topic of the FOGLI. In the next 30 minutes, the students in the treatment group worked with their peers (3 to 4 students) on our tutorial, while the students in the control group worked with their peers on the standard material. Students in both groups were encouraged to discuss with their partners while doing the exercises. After completing each of the exercises in the tutorial or the standard material, the students in the control group were provided with a printed solution of the exercise they had just completed. Students then discussed with their partners about the printed solution and compared it with their own solutions. Students might also ask the facilitators to clarify information in the printed solution. The students in the treatment group were required to check-in with the facilitator after they had completed an exercise in the tutorial, which was a part of a larger problem. The facilitator then engaged in a short conversation with the students to elicit their ideas on solving the exercise and how they perceived the exercise as relevant to the previous exercises. The facilitator might ask some questions to help them recognize their errors if there was any, but did not tell the students the correct answer to the exercise. The exercises in our tutorial were simple enough that all students were able to solve them correctly without assistance. So the role of the facilitator in the FOGLI was mostly to ensure that the students were on the right track. In the last 15-20 minutes, students individually attempted the post-test which was the same problem as in the pre-test with minor modifications on surface features of the problem.

Table 6.1 below summarizes the similarities and differences in the experimental procedures of the control and the treatment groups.

Table 6.1 Comparison of the experimental procedures of the control and the treatment groups

Group	Treatment group (N ~ 110)	Control group (N ~ 110)
Similarities	- Students worked on the pre-test and post-test problems individually. - Students worked in small groups on the exercises in the exercise sets of the tutorials or the standard materials. - Students were asked to notify the facilitator after they had completed each exercise in the set.	
Differences	- Students worked on the tutorials. - Short conversation with facilitator after each exercise. - The facilitator elicited students' ideas and provided hints if needed, but did not tell the answer.	- Students worked on the standard materials. - Printed solution provided after each exercise. - The facilitator clarified the solutions if needed, but did not tell the answer.

6.4 Data sources and analysis

Students' worksheets of the pre-test, post-test, and the tutorial were collected. Unlike mechanics problems in the spring 2010 study in which there were two separate aspects: physics and representations, the electricity problems in the fall 2010 study did not have such discrimination. The physics of the problem must be understood in order to set up the correct integral for the problem. The integral was the mathematical representation of the physical situation described in the problem. The solution to the problem, therefore, consisted of only the integral representing the quantity being asked. The solutions to the pre-test and post-test problems in the fall 2010 study were as simple as setting up and computing an integral. For these reasons, we did not grade these problems using a rubric, but instead categorized students' solutions as correct or incorrect, depending on the correctness of the integral they set up. Students were provided the integral formulae for the kinds of integral they encountered in the problems, so we did not include students' computation of the integral in the analysis.

Because of the binomial categorization of students' solutions in the pre-tests and post-tests, and because of the purpose of testing the significance of the difference between two groups, the appropriate statistical test for our study was the Fisher's exact test (Field, 2009). The null hypothesis was that the two groups were from the same population. The errors that students made in their solutions to the pre-test and post-test of each FOGLI were also recorded. These errors were then collapsed into categories corresponding to the steps in applying the integral concept to physics problems discussed in chapter 3, i.e. errors in recognizing the need for an integral in the problem, errors in setting up the infinitesimal quantity, errors in accumulating the infinitesimal quantity, errors in computing the integral.

6.5 Results

In this section, we present the pre-test and post-test problems, the number of students having the correct and incorrect integrals in each group, and the results of the Fisher's exact test in each of the three FOGLI's. Students' errors in each category will also be discussed.

6.5.1 Results of the FOGLI session 1

6.5.1.1 Result of the pre-test and post-test

There were 107 students in the control group and 112 students in the treatment group in the FOGLI session 1. Students in both control and treatment groups worked with the same partners as in their regular studio sessions.

In the first 10 minutes of the FOGLI session, all students in both control and treatment groups worked independently on the pre-test which was a problem on finding the net electric field due to a charged rod at one end of the rod. The statement and solution of the pre-test problem are presented in Figure 6.25 and Figure 6.26 below.

In the next 30 minutes, the students in the control groups worked on the standard material 1. Students were required to notify the facilitator after they had completed each exercise. The facilitator then provided the students with the solution to the exercise they had just completed. The students in the treatment group worked on our tutorial 1. Students were asked to check-in with the facilitator after they completed each exercise. The facilitator then engaged in a conversation with the students to elicit their ideas about the exercise and provided hints to help students solve the problem if needed, but did not tell them the solution. All students in both the control and the treatment groups were able to solve the exercises. The facilitator did not have to provide any hint to help students with the exercises of the tutorial 1.

In the last 10 minutes of the FOGLI session, students in both the control and the treatment groups worked individually on the post-test, which was the same problem as the pre-test.

Figure 6.25 Pre-test problem in the FOGLI session 1

Pre-test problem – FOGLI session 1

A thin non-conducting rod is lying along the x axis with the two ends at x = 0.0 m and x = 2.0 m. The charge on the rod is distributed as per the following function:

$$\lambda(x) = \alpha x^3$$

where λ(x) is the charge density at location x; α is a positive constant.
Find the electric field due to the rod at the left end of the rod, located at x = 0.0 m.

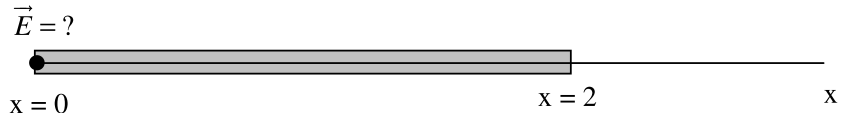
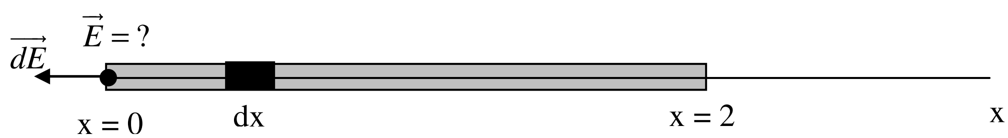


Figure 6.26 Solution to the pre-test problem in FOGLI session 1

Solution to the pre-test problem – FOGLI session 1



Consider a small segment dx along the rod, which carries a charge: $dq = \lambda(x)dx$

This charge causes at the origin an electric field: $dE = k\dfrac{dq}{r^2}$.

The distance from dq to the origin is also the location x of the charge, so:

$$dE = k\frac{dq}{x^2} = k\frac{\lambda(x)dx}{x^2} = k\frac{2x^3dx}{x^2} = 2kxdx$$

Because x is positive along the rod, λ(x) is also positive, which means that the charge on the rod is positive. So, The fields due to all charges along the rod at the origin are in the –x direction and has a magnitude:

$$E = \int dE = \int_0^2 2kxdx = 2k\left[\frac{x^2}{2}\right]_0^2 = 4k = 4x\left(9x10^9\right) = 3.6x10^{10}\,N/C$$

The integral representing the net electric field at the x = 0 end of the rod was $E_{net} = \int_0^2 2kxdx$. The number of students obtaining the correct and incorrect integral in the pre-test problem is presented in Table 6.2.

Table 6.2 Number of students getting the correct and incorrect answer in the pre-test problem of FOGLI session 1

	Correct	**Incorrect**	**Total**
Control	27	80	107
Treatment	35	77	102
Total	62	157	219

The p-value from the Fisher's exact test performed on this contingency table was p = 0.37, which meant we could not reject the null hypothesis: the two groups were from the same population. In other words, there was no statistically significant difference between the control and the treatment group in the pre-test of FOGLI session 1.

The post-test problem was identical to the pre-test problem, so the integral for the electric field was the same: $E_{net} = \int_0^2 2kxdx$. The number of students obtaining the correct and incorrect integral in the post-test problem is presented in Table 6.3.

Table 6.3 Number of students getting the correct and incorrect answer in the post-test problem of FOGLI session 1

	Correct	**Incorrect**	**Total**
Control	38	69	107
Treatment	58	54	112
Total	96	123	219

The p-value from the Fisher's exact test performed on this table was p = 0.02, so we could reject the null hypothesis: the two groups were from the same population. In other words, there was a statistically significant difference between the control and the treatment group in the post-test of FOGLI session 1.

6.5.1.2 Analysis of errors in the pre-test and post-test

Table 6.4 below summarizes the number and percentage of students making each kind of errors in the pre-test and post-test of the FOGLI session 1. Note that the number of students in each group in this table does not add up to the total number of students in that group (and the percentage does not add up to 100%) because one student might make more than one mistake and there were students who did not make any error.

Table 6.4 Number and percentage of students making each kind of error in FOGLI 1

	Control Pre-test	**Control Post-test**	**Treatment Pre-test**	**Treatment Post-test**
Not recognizing the need for an integral	31 (29%)	12 (11%)	16 (14%)	3 (3%)
Incorrect expression for the infinitesimal quantity	23 (22%)	16 (15%)	17 (15%)	5 (5%)
Incorrect accumulation of the infinitesimal quantities	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Incorrect computation of the integral	19 (18%)	12 (11%)	36 (33%)	18 (16%)

Overall, the number of students making each error decreased from the pre-test to the post-test. The students who did not recognize the use of the integral in the problem attempted to use the relation $\lambda = \frac{Q}{L}$ to find the total charge Q on the rod and plugged Q in the Coulomb's equation, or claimed that the net electric field at $x = 0$ was zero because $\lambda(0) = 0$. The students who set up incorrect expression for the infinitesimal quantity integrated $\lambda(x)$ only and plugged the result into Coulomb's equation with r as a constant. Students in both groups made a lot of errors when manipulating the integral. Some of these errors were: treating r in Coulomb's equation as a constant or having incorrect expression for r, treating λ as a constant, having wrong limits of the integral. In the pre-test and post-test problems of this FOGLI session, the electric field due to each charge element on the rod pointed in the same direction, so the net field was the algebraic sum of all the fields due to element charges. So there were no errors on accumulating the infinitesimal quantities.

6.5.1.3 Conclusion from the FOGLI session 1

The treatment group did not outperform the control group in the pre-test but they did in the post-test. The error analysis presented in Table 6.4 above shows that our tutorial 1 reduced the percentage of students making each kind of errors from the pre-test to the post-test more than the standard material 1 did. These results indicated that the students learning with our tutorial 1 improved their ability to apply the integral to problems on electric field more than the students learning with the standard material 1. In other words, students seemed to learn more from the segmented exercises than from the same (not segmented) practice exercises and solutions.

Although there were more students in our treatment group succeeded in the post-test than in the pre-test, the percentage was only about 50% in the post-test. This suggests that although our tutorial 1 has a positive impact on students' learning, it still needs to be improved to help a larger portion of students learn about integral.

6.5.2 Results of the FOGLI session 2

6.5.2 Results of the pre-test and post-test

There were 105 students in the control group and 106 students in the treatment group in the FOGLI session 2. In the first 10 minutes of the FOGLI session 2, all students in both control and treatment groups worked independently on the pre-test problem which asked for the equivalent resistance of a conductor in the shape of a truncated cone. The problem statement and solution are presented in Figure 6.27 and Figure 6.28 below.

In the next 30 minutes, the students in the control groups worked on the standard material 2 described in Figure 6.17. Students were required to notify the facilitator after they had completed each exercise. The facilitator then provided the students with the solution to the exercise, presented in Figure 6.18.

The students in the treatment group worked on our tutorial 2. Students were asked to check-in with the facilitator after they had completed each exercise. The facilitator then engaged in a conversation with the students to elicit their ideas about the exercise and provided hints to help students solve the problem if needed, but did not tell them the solution.

Similar to the FOGLI session 1, students in both the control and the treatment groups did not have significant difficulties solving the exercises in our tutorial. So the facilitator did not have to provide any hint to help students with the exercises of tutorial 2.

In the last 10 minutes of the FOGLI session, students in both the control and the treatment groups worked individually on the post-test problem, which differed from the pre-test problem in the shape of the conductor (the expression for the shape of the conductor was given).

Figure 6.27 The pre-test problem in the FOGLI session 2

Pre-test problem – FOGLI session 2

Consider a wire of length L in the shape of a truncated cone. The radius of the wire varies with distance *x* from the narrow end according to $r = a + \frac{b-a}{L}x$, where $0 < x < L$.
Derive an expression for the resistance of this wire in terms of its length *L*, radius *a*, radius *b*, and resistivity ρ.

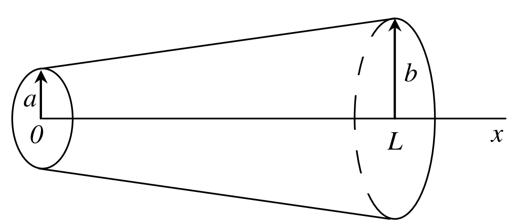


Figure 6.28 Solution to the pre-test problem in FOGLI session 2

Solution to the pre-test problem – FOGLI session 2

Consider a thin resistor whose thickness dx is small enough that over this thickness, the resistivity changes very little and hence can be considered constant, then its resistance is: $dR = \rho \frac{dx}{A} = \frac{\rho dx}{\pi \left[r(x) \right]^2}$ where r(x) is the radius of the resistor at the location x, whose function can be found using the coordinate system as drawn: $r(x) = a + \frac{b-a}{L} x$

So $dR = \rho \frac{dx}{A} = \frac{\rho dx}{\pi \left(a + \frac{b-a}{L} x \right)^2}$

Then the resistance of the whole resistor is:

$$R = \int dR = \int_0^L \frac{\rho dx}{\pi \left(a + \frac{b-a}{L} x \right)^2} = \frac{\rho}{\pi} \frac{L}{a-b} \left. \frac{1}{a + \frac{b-a}{L} x} \right|_0^L = \frac{\rho}{\pi} \frac{L}{a-b} \left(\frac{1}{b} - \frac{1}{a} \right) = \frac{\rho L}{\pi ab}$$

The correct integral for the resistance in the pre-test and post-test was $R = \int_0^L \frac{\rho dx}{\pi \left[r(x) \right]^2}$ where $r(x) = a + \frac{b-a}{L} x$ was the radius of the cross-section of the conductor at location x. The number of students getting the correct and incorrect integral in the pre-test problem is presented in Table 6.5 below.

Table 6.5 Number of students getting the correct and incorrect answer in the pre-test problem of FOGLI session 2

	Correct	**Incorrect**	**Total**
Control	8	97	105
Treatment	8	98	106
Total	16	195	211

The p-value from the Fisher's exact test performed on this table was p = 1.00, which meant that the two groups were very likely to be recruited from the same population. In other words, there was no statistically significant difference between the control and the treatment group in the pre-test of FOGLI session 2.

The post-test problem was similar to the pre-test problem except that the function for the radius of the conductor. The integral for the equivalent resistance was the same: $R = \int_0^L \frac{\rho dx}{\pi \left[r(x) \right]^2}$ where $r(x) = a + \frac{b-a}{L^2} x^2$ was the radius of the cross-section of the conductor at location x. The number of students obtaining the correct and incorrect integral in the post-test problem is presented in Table 6.6.

Table 6.6 Number of students getting the correct and incorrect answer in the post-test problem of FOGLI session 2

	Correct	Incorrect	Total
Control	33	72	105
Treatment	49	57	106
Total	82	129	211

The p-value from the Fisher's exact test performed on this table was p = 0.03, which meant that it was unlikely that the two groups were recruited from the same population. In other words, there was a statistically significant difference between the control and the treatment group in the post-test of FOGLI session 2.

6.5.2.2 Errors in the pre-test and post-test

Table 6.7 below summarizes the number and percentage of students making each kind of error in the pre-test and post-test problems.

Table 6.7 Number and percentage of students making each kind of error in FOGLI 2

	Control Pre-test	Control Post-test	Treatment Pre-test	Treatment Post-test
Not recognizing the need for an integral	12 (11%)	0 (0%)	6 (6%)	1 (1%)
Incorrect expression for the infinitesimal quantity	52 (50%)	53 (50%)	55 (52%)	27 (25%)
Incorrect accumulation of the infinitesimal quantities	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Incorrect computation of the integral	36 (34%)	17 (16%)	55 (52%)	17 (16%)

The students who did not recognize the need for an integral in the problem just plug the expression for $\rho(x)$ into the formula for the resistance of a conductor with constant resistivity and claimed that as the final answer. The most common error that led to the incorrect expression for the infinitesimal quantity was the presence of the total length L of the conductor in addition to the infinitesimal term dx (i.e. $dR = \frac{\rho L}{A} dx$ instead of $dR = \frac{\rho}{A} dx$). Since the large conductor could be considered as a series of infinite number of infinitesimally thin resistors, the equivalent resistance could be found by adding up all of the infinitesimal resistance and not the inverse of resistance as in the case of capacitance (i.e. $R = \int dR$). So there was no student making error in the accumulation step. The errors in computing the integral included incorrect variable of integration, incorrect limits of the integral, confusion between constants and variables.

6.5.2.3 Conclusion from the FOGLI session 2

The results of the FOGLI session 2 were similar to those of the FOGLI session 1: the treatment group did not outperform the control group in the pre-test but they did in the post-test. The same trend was observed in the error analysis in FOGLI session 2 as in the FOGLI session

1: fewer errors were made in the post-test than in the pre-test. This implied that students learning either with our tutorial 2 or the standard material 2 had improved their ability to apply the integral to physics problems. Larger reduction was also observed in the treatment group than in the control group. These results indicated that our tutorial 2 helped students improve their ability to apply the integral to electricity problems more than the standard material 2 did. A possible explanation for this result was that as students worked on short, simple exercises and did not know the final goal of the sequence (i.e. they did not know what exercise would come next), they might attend more closely to every details of each exercise. This helped them learn the meaning of each term in the equations being used and the procedure being done. On the other hand, as students attempted the problem as a whole and then read the printed solution, they already had a final goal to look forward to. That might make them overlook necessary information and skim through the solution to reach the final result quickly without noticing every details of the solution. In the particular problem of the FOGLI session 2, as students in the treatment group worked through the sequence of exercises, they were led from the case when there were a few capacitors to the case when there was infinite number of capacitors. Through this process, students learned how the sum of a few capacitance became the integral of capacitance of each individual capacitor, and what the infinitesimal term dx meant (i.e. the separation Δx between the plates of the capacitor became dx when the number of capacitors became infinite). On the other hand, as students in the control group attempted the problem, they started with the equation for capacitance of a capacitor with small separation and had to find the capacitance of a capacitor with large separation, with no hints on the intermediate steps or procedures. When these students read the solution, since they already had the final goal of finding the capacitance, they might just look at the equations and the final answer without careful investigation of the strategy described in the text of the solution. So these students might not get all of the information presented in the solution as they read it.

Although there were more students in our treatment group succeeded in the post-test than in the pre-test, the percentage of students getting the correct answer was still less 50% in the post-test. This result was similar to the FOGLI session 1, which implied that our tutorial needed to be strengthened to make a positive effect on a larger population of students.

6.5.3 Results of the FOGLI session 3

6.5.3.1 Results of the pre-test and post-test

There were 100 students in the control group and 103 students in the treatment group in the FOGLI session 3. In the first 10 minutes of the FOGLI session 3, all students worked independently on the pre-test problem which asked for the net current in a wire made of a cylindrical wire of radius R_1 at the core, which was coated by a coaxial conducting cylindrical shell of inner radius R_1 and outer radius R_2. The core and the shell were carrying electric current with different current densities and in opposite directions. The problem statement and solution are presented in Figure 6.29 and Figure 6.30 below.

In the next 30 minutes, the students in the control groups worked on the standard material 3 described in Figure 6.21 and Figure 6.23. Students were required to notify the facilitator after they had completed each exercise. The facilitator then provided the students with the solution to the exercise, presented in Figure 6.22 and Figure 6.24.

The students in the treatment group worked on our tutorial 3 described in Figure 6.19 and Figure 6.20. The exercises in the tutorial and the standard material were simple, so all students in both the control and the treatment groups were able to solve them correctly without assistance.

In the last 10 minutes of the FOGLI session, students in both the control and the treatment groups worked individually on the post-test problem which was identical to the pre-test problem.

Figure 6.29 The pre-test problem in the FOGLI session 3

Pre-test problem – FOGLI session 3

A long, straight wire has a radius R_1 and carries a current with current density $j_1(r) = \alpha r^2 \ (r \le R_1)$ going into the page. This wire is coated by a coaxial cylindrical shell which has inner radius R_1 and outer radius R_2, and carries a current with current density $j_2(r) = \beta r \ (R_1 \le r \le R_2)$ going out of the page. Find the net current in the wire.

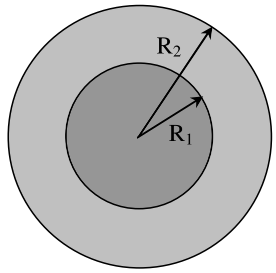


Figure 6.30 Solution to the pre-test problem in the FOGLI session 3

Solution to the pre-test problem – FOGLI session 3

Consider an infinitesimally thin ring which has inner radius *r* and thickness *dr* small enough that the current density *j(r)* is almost constant across the cross section of this ring.
The cross sectional area of this ring is:

$dA = \pi(r+dr)^2 - \pi r^2 = \pi\left(r^2 + 2r.dr + dr^2\right) - \pi r^2 = 2\pi r dr$ (note that *dr* is very small so $dr^2 \approx 0$)

The current through this cross sectional area is: $dI = j(r)dA$

The current in the inner shell (going into the page) is:

$$I_{inner} = \int dI = \int_0^{R_1} \alpha 2\pi r^3 dr = \alpha 2\pi \left.\frac{r^4}{4}\right|_0^{R_1} = \frac{\alpha\pi}{2} R_1^4$$

The current in the outer shell (going out of the page) is:

$$I_{outer} = \int dI = \int_{R_1}^{R_2} \beta 2\pi r^2 dr = \beta 2\pi \left.\frac{r^3}{3}\right|_{R_1}^{R_2} = \beta 2\pi\left(\frac{R_2^3 - R_1^3}{3}\right) = \frac{2}{3}\beta\pi\left(R_2^3 - R_1^3\right)$$

The net current in the wire is then:

$$I_{net} = \left|I_{inner} - I_{outer}\right| = \left|\frac{\alpha\pi}{2}R_1^4 - \frac{2}{3}\beta\pi\left(R_2^3 - R_1^3\right)\right|$$

The number of students getting the correct and incorrect results in the pre-test problem is presented in Table 6.8 below.

Table 6.8 Number of students getting the correct and incorrect answer in the pre-test problem of FOGLI session 3

	Correct	Incorrect	Total
Control	33	67	100
Treatment	42	61	103
Total	75	128	203

The p-value from the Fisher's exact test performed on this table was p = 0.31, which meant that there was no statistically significant difference between the control and the treatment group in the pre-test of FOGLI session 3.

The number of students obtaining the correct and incorrect integral in the post-test problem is presented in Table 6.9.

Table 6.9 Number of students getting the correct and incorrect answer in the post-test problem of FOGLI session 3

	Correct	Incorrect	Total
Control	42	58	100
Treatment	55	48	103
Total	97	106	203

The p-value from the Fisher's exact test performed on this table was p = 0.12, which meant that there was no statistically significant difference between the control and the treatment group in the post-test of FOGLI session 3.

6.5.3.2 Analysis of errors in the pre-test and post-test

Table 6.10 below summarizes the number of students making each kind of error in the pre-test and post-test problems.

Table 6.10 Number and percentage of students making each kind of error in FOGLI 3

	Control Pre-test	Control Post-test	Treatment Pre-test	Treatment Post-test
Not recognizing the need for an integral	20 (20%)	7 (7%)	27 (26%)	8 (8%)
Incorrect expression for the infinitesimal quantity	30 (30%)	18 (18%)	38 (37%)	17 (17%)
Incorrect accumulation of the infinitesimal quantities	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Incorrect computation of the integral	49 (49%)	40 (40%)	37 (36%)	28 (27%)

Some of the students who did not use integral in the pre-test and post-test problems in FOGLI the session 3 just plugged the expression of current density into the equation $I = jA$ where A was the total area of the wire, and claimed the final expression as the final answer. Some other students plugged the radius of the wire into the expression for the current density and multiplied by the total area (i.e. $I = \left[j(R) \right] \pi R^2$). The incorrect expressions for the infinitesimal quantity that students set up were $A \cdot j(r)$ or $A \cdot j(r)dr$ instead of $j(r)dA$. The errors in computing the integral included: confusion between variables and constants, incorrect expression for the infinitesimal area dA, incorrect limits of the integral, inappropriate variable of integration.

6.5.3.3 Conclusion from the FOGLI session 3

The results of the Fisher's exact tests indicated that there was no statistically significant difference between the control and the treatment groups in both the pre-test and the post-test. The error analysis showed a reduction in the percentage of students making each kind of error in both groups. Although a slightly larger reduction was observed in the treatment group than in the control group, that was not sufficient to make a significant difference in the number of students getting the correct answer in the two groups.

Taking a closer look at the errors students made, we might find a possible explanation for the insignificant result in this FOGLI session. There were three errors that many students in both groups made. First, students took the integral of $j(r)$ itself and multiplied the result with the total area A of the wire, i.e. $I = A\int j(r)\,dr$ while the correct integral must be $I = \int j(r)\,dA$. Twenty eight students in the control group and 33 students in the treatment group made this error in the pre-test, while there were still 18 students in the control group and 15 students in the treatment group made this error in the post-test. The second error was that students had incorrect expression for the infinitesimal area dA. In the pre-test, 34 students in the control group and 27 students in the treatment group were not able to write the correct expression for dA. In the post-test, there were still 22 students in the control group and 17 students in the treatment group made this error. The third error was the incorrect limit of the integral. Ten students in each group made this error in the pre-test, while these numbers in the post-test were 11 students in the control group and 10 students in the post-test. We see that there were many students making these three errors and there was not much improvement on these errors between the pre-test and post-test in both groups. Except for the first error which was similar to the error in the infinitesimal quantity in FOGLI session 2, the other two errors were closely related to the fact that students were integrating with respect to area. We observed from the individual interviews in the fall 2009 study that most of the students were having significant difficulty making sense of an integral with respect to area, i.e. $\int j(r)\,dA$. It seemed that students were so familiar with integrating with respect to positional variables such as dx, dr, $d\theta$, … so it did not make sense to them to integrate with respect to area dA. Almost all of the students in the fall 2009 interviews were not able to interpret the meaning of dA and hence, they failed to derive an expression for dA and to determine the limits of integration. The same difficulties were observed in the FOGLI session 3 of the fall 2010 study. The fact that there was not much improvement on these errors between the pre-test and post-test indicated that our tutorial 3 as well as the standard material 3 seemed to be insufficient in training the students to work with this kind of integral. Therefore, there was no significant improvement in the results of the pre-test and post-test.

6.6 Conclusion

In this study, we created three tutorials to facilitate students' learning to solve electricity problems involving the integral concept. Each tutorial consisted of a sequence of exercises and a protocol for the conversation between the facilitator and the students after they had completed each of the exercises. The exercises in the sequence were designed to lead students from a simple case of adding up discrete quantities to the more abstract case of integrating continuous quantities. Working through the sequence of exercise, students might learn how an individual object became an infinitesimal part of a larger object, and how a sum became an integral. Through this process, students learned or reinforced their knowledge of the accumulative nature of the integral and the meaning of the integrand as well as the infinitesimal term. All of the exercises in the tutorials were pretty simple so all groups were able to get the correct answers. Therefore the conversation between the facilitator and the students that took place after each exercise was mostly to elicit students' ideas about the exercise and how they perceived the exercises to be related. The facilitator did not have to provide hints to help students solve the exercises in the tutorial.

We conducted focus group learning interviews (FOGLI's) to test the effectiveness of our tutorials in comparison to standard materials. The standard materials consisted of typical end-of-chapter exercises and solutions that covered the same concepts as the exercises in the tutorials. We found that the first two tutorials on the electric field and the resistance problems improved students' ability to apply the integral into physics problems significantly more than the standard materials did. However, the percentage of students being able to obtain the correct answer in the post-test in these two FOGLI sessions was still less than 50%. So our tutorials must be strengthened to facilitate more students to learn about integration in electricity problems. Our third tutorial did not provide promising results as the other two. The major reason for this might be that the students were unfamiliar with the type integral in the third tutorial. This tutorial needed to be revised to teach students even more about integrating with respect to area.

6.7 Limitations and future work

This study overcame the limitation encountered in the spring 2010 study on the sample sizes. Our tutorials in the fall 2010 were administered to all students (about 200+) enrolling in the course. However, because the tutorials were administered as part of the studio sessions of the course, they were limited on the amount of time and hence the amount of training via the tutorials. Each tutorial in this study was much shorter than the tutorials in the spring 2010 study. This was the major limitation of our tutorials in this study.

Although our tutorials had provided some promising results, there was still about half of the students who were not able to learn effectively from our tutorials. So we plan on improving the training power of the tutorials by extending its length and revising the exercises to better address students’ difficulties.

Chapter 7 - Investigating the Development of Students' Application of Mathematical Concepts in Physics Problem Solving – Two Case Studies

7.1 Introduction

In the spring 2009 semester, we conducted 80 interviews with 20 students on mechanics problems involving the integral and the area under the curve concepts. These interviews provided us with a close look at students' difficulties in applying these concepts to mechanics problems and the hints that might help students overcome those difficulties, as presented in Chapter 3. The findings from these interviews also constituted the basis for the development of tutorials to facilitate students' learning apply the integral and the area under the curve concepts in mechanics problems, as presented in Chapter 5.

The fact that we interviewed the same students for several times on the same concepts over a semester also makes it possible to trace the conceptual development of individual students over time. In this chapter, we will exploit the longitudinal aspect our study described in Chapter 3. The transfer in pieces framework by Wagner (Wagner, 2006), which is introduced in sub-section 2.3.4 of the literature review in this dissertation, will be employed to interpret and trace the development of an individual student as he progresses through our interviews. In this framework, Wagner introduced the term concept projection, which was "a specific combination of knowledge resources and cognitive strategies used by an individual to identify and make use of a concept under particular contextual conditions." (Wagner, 2006, p. 10)

In order to determine whether a concept is applicable in a certain context, a student has to activate and combine the relevant knowledge resources from his/her knowledge structure. These knowledge resources constitute a knowledge base on which the student bases his/her reasoning about the applicability of the concept. Different students may activate different resources to determine whether a concept is applicable in a certain situation. The activation of an inappropriate resource or the missing of an appropriate resource in the knowledge base may lead the student to perceive a concept as applicable while in fact it is not, and vice versa.

We define three terms that will used in the following analysis: knowledge frame, knowledge base, and concept projection. The *knowledge frame* refers to how the student frames the situation by putting together the relevant information read out from the situation. For example, when an expert is asked to solve one of the algebraic problems of our interviews, he is likely to frame the problem as follows: I have a non-constant force presented in algebraic representation and I must calculate the work done by that force. I know that the total work is the sum of the incremental work on small segments of the path, which becomes an integral when the segments become very small. Mathematically, this is done by integrating $F \cdot ds$. A schematic representation of this knowledge frame is presented in Figure 7.1. A *knowledge base* is a collection of knowledge resources that a student puts together in order to reason about the applicability of a concept in a certain situation. A *concept projection* is a "particular set of knowledge elements and readout and reasoning strategies that permit a concept to be perceived as applicable to a situation having particular characteristics or affordances." Affordance, according to Wagner, is "the support offered by any aspect of a situation that the individual perceives as being relevant to the (problem-solving) activity at hand." (p. 11)

In this chapter, we present two case studies. The first case study investigates the knowledge resources that a student – Alex (pseudonym) – activated to recognize the applicability of the integral concept in calculating the work done by non-constant forces in the algebraic problems of interviews 2, 3, and 4 in the Spring 2009 study. The research questions in this case study are:

- What resources did Alex use to recognize the applicability of the integral concept in calculating the work?
- How did Alex's concept projection of the integral concept change as he progressed through the interviews?

The second case study investigates the knowledge resources that another student – Eric (pseudonym) – activated to recognize the applicability of the area under the curve concept in calculating the work done by non-constant force in the graphical problems of interviews 2, 3, and 4 of the Spring 2009 study. The research questions in this case study are:

- What resources did Eric use to recognize the applicability of the area under the curve concept in calculating the work?
- How did Eric's concept projection of the integral concept change as he progressed through the interviews?

7.2 Case study #1 – Algebraic Representation

7.2.1 The interview problems

The algebraic problems in interview 2 (Figure 7.2) and interview 3 (Figure 7.4) involved finding the work done by non-constant forces. The force functions $F(x)$ were provided as algebraic expressions of x. Students had learned from the lecture that the formula for the work done by a constant force $F.d$ became the integral $\int F(x)dx$ when the force was not constant over the whole distance, but there were no homework or exam problems involving non-constant forces. So, students did not have any formal practice in finding the work using the integral prior to our interviews.

A student is most likely to recognize the use of the integral concept in calculating the work done by a force if he is able to activate all of the following knowledge resources:

- The total work W is the sum of incremental work dW on small segments of the trajectory, which in turn becomes an integral of the infinitesimal work $\int dW$ when the segments are infinitesimally small.
- The work equals the integral of force. (Note: There is no mention here of what the variable of integration is.)
- The integral must be that of the product of force and the length of a segment, i.e. the integral of $F(x)dx$.

These resources constitute a complete knowledge base for the concept projection of the integral concept for calculating work. Although these resources seem to be overlap with each other, they are treated as separate resources in our analysis for two reasons.

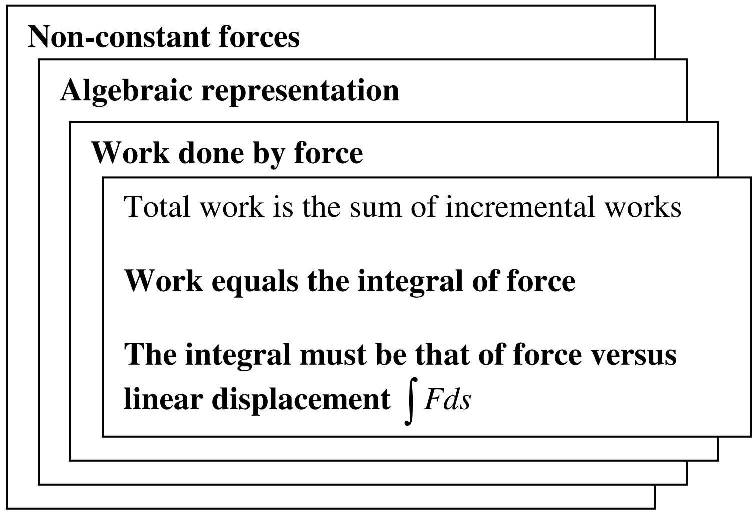
First, students might activate just one of these resources, which might lead them to an inappropriate application of the integral concept in the problem. The activation of the first resource – the total work is the sum of incremental works on small segments of the trajectory – is productive only if students know the formula for the infinitesimal work on each segment. However, our study on students' understanding of integration in mechanics presented in Chapter 3 of this dissertation indicates that students usually do not know the formula for the infinitesimal quantity. So, it is unlikely that this resource could be spontaneously activated by the students when solving the problems. The second resource – the work equals the integral of force – is applicable only to the problems in which the forces are provided as functions of linear displacement. If the force is given as a function of time, for instance, then the integral of force does not represent the work, but instead the impulse of the force during a certain time interval. The second resource needs to be complimented by the third resource to make a complete knowledge set for applying the integral concept to calculating the work done by a force.

The second reason for treating the resources as separate is that some of these resources are explicitly presented to the students either in lecture or in the text, so students might just recall them without understanding their underpinnings. Oftentimes, such a resource is an equation that is provided to the student on an equation sheet during an exam. For example, the knowledge that work equals integral of the force was explicitly taught in the course from which our interviewees were recruited. Hence, when a student talked about "work equals the integral of force," it was possible that he was just recalling what he had learned from the lecture.

The algebraic problem of interview 4 (Figure 7.6) also involved calculating the work done by a non-constant frictional force, but the force function given was $F(\theta)$ where θ was the angular displacement of the object on a circular track. The integral $\int F(\theta)\,d\theta$ is the sum of the product of force and angle, but this is not equal to the work done. The total work done is the sum of the works on small segments of the track, which is the product of the force and the length of a segment of the track. This sum is the integral $\int F(\theta)\,ds$ when the length ds of each segment became infinitesimally small. This integral could be written in terms of θ as $\int F(\theta)\,R d\theta$, because of the relation $ds = Rd\theta$. To solve this problem, students must have a concept projection of the integral which consists of all three knowledge resources as mentioned above. Without the last resource, students might claim any integral of force, such as the integral with respect to angle $d\theta$, as the value of the work done by the force.

Figure 7.1 below shows a representation of a possible knowledge frame that is likely to be used by an expert to calculate the work done by non-constant forces in the algebraic problems of interviews 2 through 4. We have adapted Wagner's schematic representation in which the knowledge resources used by an individual to frame the problem at hand were highlighted.

Figure 7.1 A possible expert's knowledge frame for calculating the work done by a non-constant force when the force is provided in graphical representation.



We will now analyze the performance of a student, Alex (pseudonym), as he calculated the work done by non-constant forces using integrals. This analysis will reveal his concept projections for the integral in the work problems and their relation with his success or failure on the tasks.

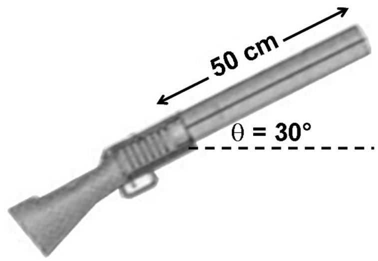
7.2.2 Results of case study #1

7.2.2.1 Interview 2 – Algebraic problem

The algebraic problem in this interview is presented in Figure 7.2 below. The velocity of the bullet at the end of the muzzle could be obtained by using the work-kinetic energy theorem, in which the work done by the spring force is calculated by integrating the force function with respect to linear displacement. In this interview, Alex did the algebraic problem prior to the graphical problem. He was able to set up the equation for the work-kinetic energy theorem quite easily. Then he attempted to find the work done by the spring force.

Figure 7.2 The algebraic problem in interview 2

A 0.1 kg bullet is loaded into a gun (muzzle length 0.5 m) compressing a spring to a maximum of 0.2 m as shown. The gun is then tilted at an angle of 30° and fired.



The only information you are given about the gun is that the barrel of the gun is frictionless and that the gun contains a non-linear spring such that when the gun is held horizontally, the net force F (N) exerted on a bullet by the spring as it leaves the fully compressed position varies as a function of the spring compression x (m) as given by:

$$F = 1000x + 3000x^2$$

What is the muzzle velocity of the bullet as it leaves the gun, when the gun is fired at the 30° angle as shown above?

Interviewer: *So how do you find the work done by the spring force?*

Alex: *Work is force time distance* [writes $F \cdot d$]

Interviewer: *What value of force would you use to plug in?*

Alex: *Umm … this one* [points at $F = 1000x + 3000x^2$]

Interviewer: *That's not a value, it's a function. That means for each value of x you have a different value of force.*

Alex: *Oh .. okay … force is not constant … so I have to do the integral then.*

Interviewer: *What quantity does the integral represent here?*

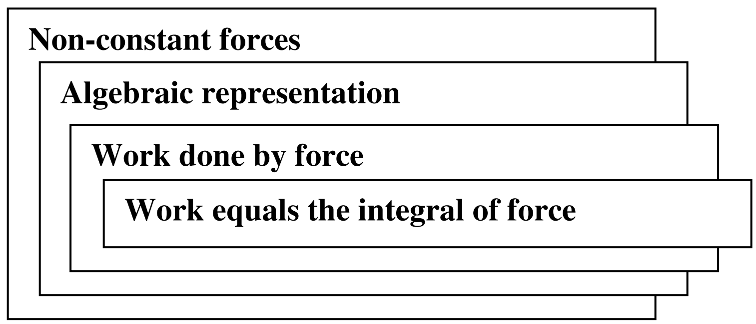
Alex: *Integral of force is work.*

Interviewer: *Okay, let's do it.*

Alex first attempted to use the equation for the work done by a constant force ($W = F \cdot d$) to calculate the work done by the spring. Upon realizing that the spring force was not constant, he was able to recognize that the "*the integral of force is work.*" In the language of the transfer in pieces framework, we could say that Alex started out not having a concept projection for the integral concept for finding the work done by a non-constant force.

As he was guided to think of the non-constant nature of the force, he was able to recall the knowledge that "i*ntegral of force is work"* which then allowed him to see the use of the integral in finding the work. He then had a concept projection of the integral concept which consisted of only one knowledge resource: the work equaled the integral of force. The knowledge frame for Alex's concept projection in interview 2 is presented in Figure 7.3.

Figure 7.3 Alex's knowledge frame that guides his thinking in the algebraic problem in interview 2

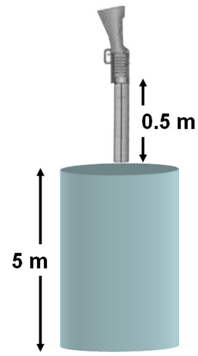


7.2.2.2 Interview 3 – Algebraic problem

The algebraic problem in this interview is presented in Figure 7.4 below. Similar to interview 2, the algebraic problem in interview 3 also requires the application of the work-kinetic energy theorem with the work done by a frictional force being calculated by an integral of force. In this interview, Alex also attempted the algebraic problem prior to the graphical problem. He had no difficulty setting up the equation for the work-kinetic energy theorem as well as recognizing the integral concept for finding the work done by the frictional force.

Figure 7.4 The algebraic problem in interview 3

A 0.1 kg bullet is loaded into a gun compressing a spring which has spring constant k = 6000 N/m. The gun is tilted vertically downward and the bullet is fired into a drum 5.0 m deep, filled with a liquid.



The barrel of the gun is frictionless. The frictional force F(N) provided by the liquid changes with depth x(m) as per the following function.

$$F = 10x + 0.6x^2$$

The bullet comes to rest at the bottom of the drum
What is the spring compression x?

Alex: *I have an equation for the frictional force, so I'm gonna take integral of it.*

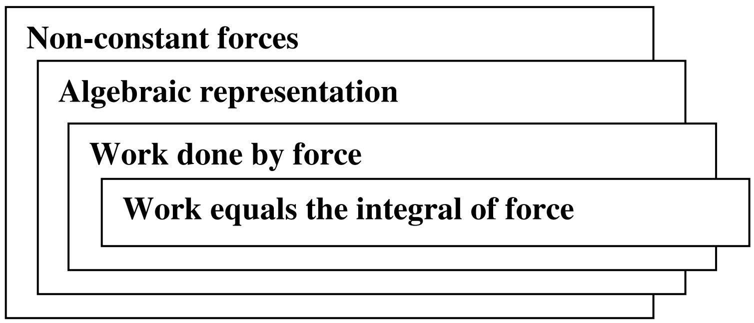
Interviewer: *What does that integral represent?*

Alex: *Integral is the work by the frictional force.*

In this problem, Alex no longer struggled with finding the work done by the frictional force. He easily recognized that he had to integrate the force function by invoking the knowledge resource that "*integral is the work.*" This was the only knowledge resource that Alex used to cue integration in this problem. Although he did not mention that the integral must be of the force function with respect to displacement, he wrote down the integral $\int_0^5 F(x)\,dx$ on his worksheet, which implied that by "integral", he meant "integrating $F(x)$ with respect to x." This integral was actually the correct integral for the work done by the resistance force of the liquid. So the only knowledge resource that "the work equals the integral of force" was adequate for Alex to perceive the integral concept as applicable in this problem. In other words, the concept projection of the integral concept which consists of only the second resource in the expert's frame in Figure 7.1 was productively used by Alex in this problem.

Figure 7.5 below describes the knowledge frame that guided Alex's thinking about the integral concept in this problem. This is identical to the frame he used in the algebraic problem in interview 2.

Figure 7.5 Alex's knowledge frame that guides his thinking in the algebraic problem in interview 3

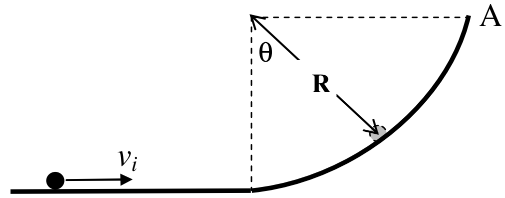


7.2.2.3 Interview 4 – Algebraic problem

The statement of the algebraic problem in interview 4 is presented in Figure 7.6 below. Similar to the algebraic problems in interviews 2 and 3, this problem also involved the work-kinetic energy theorem in which the work done by the frictional force was calculated from the force function. However, the frictional force in this problem was provided as a function of angular displacement instead of linear displacement as in the other two interviews. The integral of force, $\int_0^{\pi/2} F(\theta)\,d\theta$, was therefore no longer the value of the work. The correct integral of work in this problem must be $\int_0^{\pi R/2} F(\theta)\,ds$ in which $ds = Rd\theta$ was the length along the track of radius R spanning the angle $d\theta$. In the graphical problem that Alex attempted prior to this algebraic problem, he needed hints from the interviewer to recognize that the area under the curve was not yet the value of work and that he had to multiply it by the radius of the track. As he moved on to the algebraic problem, he easily recognized the use of the integral in calculating work by relating this problem with the graphical problem. However, there was a mismatch between the integral and the area that made his attempt a failure.

Figure 7.6 The algebraic problem in interview 4

A sphere radius r = 1 cm and mass m = 2 kg is rolling at an initial speed v_i of 5 m/s along a track as shown. It hits a curved section (radius R = 1.0 m) and is launched vertically at point A. The rolling friction on the straight section is negligible.



The magnitude of the rolling friction force F_{roll} (N) acting on the sphere varies as angle θ (radians) as per the following function

$$F_{roll}(\theta) = -0.7\theta^2 - 1.2\theta + 4.5$$

What is the launch speed of the sphere as it leaves the curve at point A?

Interviewer: *So what are the similarities and differences between this problem and the graphical one?*

Alex: *They are the same, I would say. But now you have an equation instead of a graph, so I have to do an integral instead of the area under the curve.*

Interviewer: *What will your integral look like?*

Alex: [writes down $\int F(\theta)$]

Interviewer: *What variable are you taking integral with?*

Alex: *Umm … θ maybe.*

Interviewer: *So you must have the differential term … I mean $d\theta$ … to indicate that.*

Alex: *Okay* [writes $\int_0^{90} F(\theta)\, d\theta$, starts computing the integral and gets 267.5]

Interviewer: *What is that number?*

Alex: *It's work.*

Interviewer: *What is the unit of that number?*

Alex: *Unit of work is Joule.*

Interviewer: *Let's look carefully at what quantity you are integrating. You integrate force time angle, so what is the unit then?*

Alex: *Oh … so Newton time … degree?*

Interviewer: *Yes, Newton time degree. But what is the unit of work?*

Alex: *Joule.*

Interviewer: *But what is one Joule?*

Alex: *One Joule is one Newton time meter. So I have to convert degree to meter somehow.*

Interviewer: *How would you do that?*

Alex: *Umm … the sphere travels a quarter of the circle, so the angle is 90 degrees … and the length is … a quarter of a circumference is …* $2\pi R$ *over 4.*

Interviewer: *So what is your conversion factor?*

Alex: *90 degrees over* $2\pi R$ *over 4 meter* [writes $\frac{90}{\frac{2\pi R}{4}} = \frac{180}{\pi R}$ degree over meter]

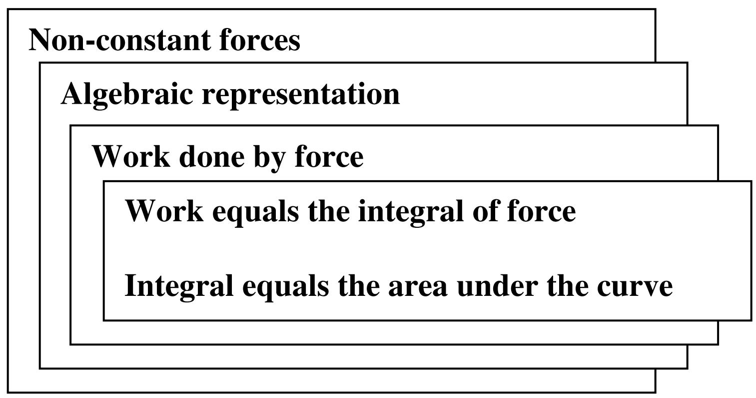
Interviewer: *Are you doing it the other way around?*

Alex: *Oh yes, I need meter over degree … so it would be* $\frac{\pi R}{180}$ [does the unit conversion and gets 4.67 Joules]

Right at the beginning of the problem, by comparing the algebraic problem with the graphical one, Alex easily recognized that he had to integrate the force function. His written integral indicated that by "*do an integral*," Alex meant to integrate $\int F(\theta)\, d\theta$, i.e. integrate the given force function with respect to its variable. By ignoring the term $d\theta$, Alex demonstrated his lack of understanding of the components of an integral, which disabled him from thinking about the total work as the sum of infinitesimal work. Further, he did not even have the correct expression for the infinitesimal work. Consequently, this prevented him from noticing that the expression inside the integral must be $F(\theta)\, ds$, i.e. invoking the third knowledge resource in the expert's knowledge frame for the integral concept for calculating the work. In fact, Alex used only one knowledge resource from this frame – integral of force was work – together with another knowledge resource that "the area under the curve equaled the integral" to decide on the applicability of the integral concept in this problem.

Figure 7.7 below shows the knowledge frame that Alex used in this problem. As analyzed earlier, missing the third knowledge resource from the expert's knowledge frame led students, such as Alex, to perceive any integral of force as the value of work. Alex's concept projection for the integral concept, which consisted of only the second resource in the expert's frame, spanned only the problems in which the force functions $F\left(x\right)$ were given. So it led to a failure when Alex attempted to use it in interview 4 when the force was not given as a function of linear displacement.

Figure 7.7 Alex's knowledge frame that guides his thinking in the algebraic problem in interview 4



7.2.3 Summary of Case Study #1

At the first time Alex encountered the task of calculating the work done by a force using the integral concept (i.e. in interview 2), he did not have a concept projection of the integral concept so he struggled to find the work done by the force. Upon being guided to think about the non-constant nature of the force, he was able to activate the resource that "the integral of force was the work," which was true for the function provided in that interview. This knowledge resource also constituted Alex's concept projection for the integral concept for finding the work in the algebraic problem of interview 3. He carried the same concept projection into interview 4, where he combined with it one more knowledge resource that "the area under the curve equaled the integral." This development in the knowledge base led to the extension of the span of Alex's concept projection: it then spanned not only problems in which the force was given as a function of linear displacement but also as a graph of force versus linear displacement. However, the knowledge base for Alex's concept projection in interview 4 was missing an important knowledge resource: "the integral must be that of $F(x)dx$." Without this resource, his concept projection did not span the type of problems in which the force function provided was not that of force versus linear displacement. This explained Alex's failure when he used his concept projection in the algebraic problem of interview 4.

We answer our the research questions in this study as follows:

- What resources did Alex use to determine the applicability of the integral concept in calculating the work?

Most of the times in our interviews, Alex activated the knowledge resource that "the integral of force was the work" to determine use of the integral concept in calculating the work done by non-constant forces. In interview 4, Alex did the graphical problem before the algebraic problem, so he also activated another resource – "the area under the curve equaled the integral" – as a cue for using integral to calculate the work. However, he failed activate the resource that "the integral must be that of $F(x)dx$" so he did not recognize that the integral of force in interview 4 did not yield the value of work.

- How did Alex's concept projection of the integral concept change as he progressed through the interviews?

The first time Alex encountered the algebraic problem, he did not have a concept projection for the integral concept in calculating the work. In most of the later instances in the interviews, Alex attended to the given function and the knowledge resource that "the integral of force was the work" to determine that the integral concept is applicable in the problems. In interview 4, he attended to these information and also the integral-area relation to perceive the integral concept as applicable. This was a development in his concept projection of the integral concept which allowed him to perceive the graphical and the algebraic problems to be similar. His concept projection of the integral at that time spanned the problems in which the function of force was given in not only the algebraic representation but also the graphical representation. However, Alex did not attend to the variable of the force function provided, so his concept projection did not span the type of problems in which the force provided was not a function or a graph of force versus a variable other than linear displacement.

7.3 Case study #2 – Graphical representation

7.3.1 The interview problems

The graphical problems presented to students in interviews 2 and 3 (Figure 7.9 and Figure 7.11) involved finding the work done by non-constant forces from the graphs of force versus linear displacement. Students had learned from the lecture that the work done by a force equaled the area under the curve of force versus displacement, but there were no homework or exam problems in which this knowledge was required. So students did not have any prior experience of finding work using the area under the curve prior to our interviews.

A student is most likely to recognize the use of the area under the curve concept in calculating the work done by a force if he possesses the following knowledge resources:

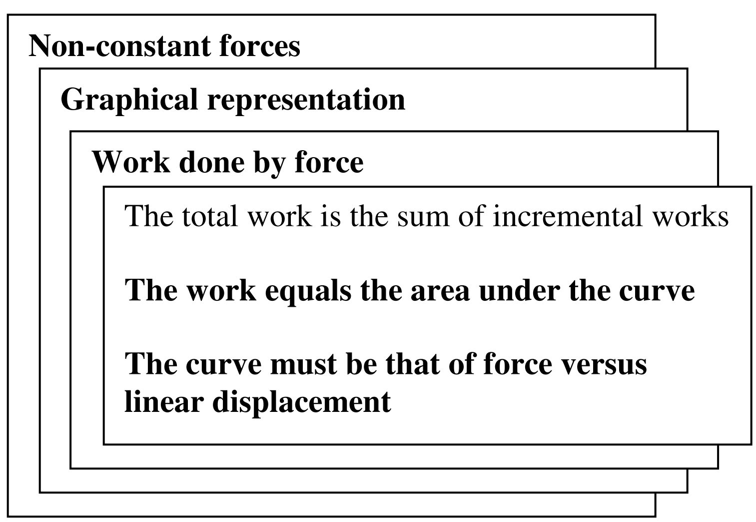
- The total work is the sum of incremental works on small segments of the trajectory;
- The work equals the area under the curve;
- The curve must be on a graph of force versus linear displacement, i.e. the graph of $F(x)$ vs. x.

These resources constitute a complete knowledge base for the concept projection for the area under the curve method for calculating the work. These resources could be combined to make a more complete resource: the total work is the sum of incremental works on small segments of the trajectory which was equivalent to the sum of all incremental areas under the curve of force versus linear displacement, i.e. the total area under the curve of $F(x)$ vs. x. However, we treat them as separate resources in our analysis because of the same two reasons mentioned in the case study 1. First, students might activate just one of the resources when solving a problem. The activation of only the knowledge resource that "the work equals the area under the curve" might lead students to perceive the area under the curve of any graph of force as the work done by that force. For example, the area under the curve of force versus time might be claimed as the value of work while it is in fact the value of impulse. Therefore, the concept projection that allows the recognition and appropriate application of the area under the curve concept in finding the work done by a force must include the knowledge resource that "the graph must be that of force versus linear displacement." Second, students might activate a knowledge resource that had been taught explicitly in the lecture without understanding its underpinnings or underlying assumptions. The idea that "the work equals the area under the curve" was taught explicitly in the lecture, so when students talked about finding work using the area under the curve, it was possible that they were recalling this knowledge without understanding the underlying accumulation process.

The graphical problem of interview 4 also involved calculating the work done by friction force by the graphical method, but the graph given was that of force versus angular displacement of the object on a circular track. The area under the curve was then the sum of the product of force and angle, which was not the value of work. To solve this problem, students must have a concept projection for the graphical method which consists of all three knowledge resources mentioned above. Without the last resource, students would claim that the area under the curve was the value of the work done by friction.

Figure 7.8 below shows a possible knowledge frame that is likely to be used by an expert to calculate the work done by non-constant forces in the graphical problems in our interviews.

Figure 7.8 A possible expert's knowledge frame for calculating work in the graphical problems in our interviews



We will now analyze the performance of a student, Eric (pseudonym) as he calculated the work done by non-constant forces using the area under the curve concept. This analysis will help identify Eric's concept projections for the method and their relation with his success or failure on the task.

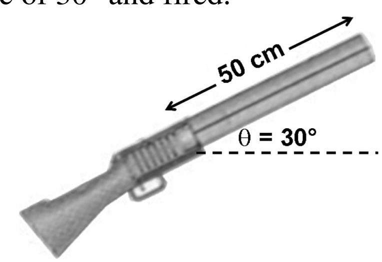
7.3.2 Results of case study #2

7.3.2.1 Interview 2 – Graphical problem

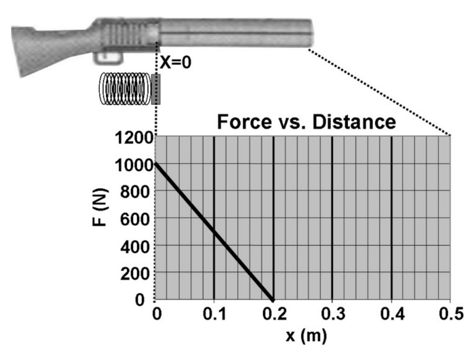
The graphical problem in this interview is presented in Figure 7.9 below. The strategy for finding the velocity of the bullet is the same as in the algebraic problem, except that the work done by the spring force could be now calculated using the area under the curve of force (i.e. the triangular section on the graph). With a few hints given by the interviewer, Eric was able to set up the equation for the work-kinetic energy theorem. Then he started struggling with the graph.

Figure 7.9 The graphical problem in interview 2

A 0.1 kg bullet is loaded into a gun (muzzle length 0.5 m) compressing a spring as shown. The gun is then tilted at an angle of 30° and fired.



The only information you are given about the gun is that the barrel of the gun is frictionless and when the gun is held horizontally, the net force F (N) exerted on a bullet by the spring as it leaves the fully compressed position varies as a function of its position x (m) in the barrel as shown in the graph below.



What is the muzzle velocity of the bullet as it leaves the gun, when the gun is fired at the 30° angle as shown above

Eric: *We are not given k and x though.*

Interviewer: *What do you need k and x for?*

Eric: *Because work done by the spring is* $\frac{1}{2}kx^2$.

Interviewer: *Yes, but can you think of a way to find work done by the spring without knowing k and x explicitly?*

Eric: *… I don't know.*

Interviewer: *The main source of information about the spring is the graph, so let's explore it. What information can you extract from this graph?*

Eric: *The slope.*

Interviewer: *What does the slope tell you about the spring?*

Eric: *Um … slope is … maybe k … or … work …*

Interviewer: *What other information can you read out from the graph?*

Eric: *Ummm … I don't know … I'm not good at graphs though.*

Interviewer: *Okay. Did you learn how to find work from graph in the lecture?*

Eric: *Ummm … oh, I can find area under the curve … yeah.*

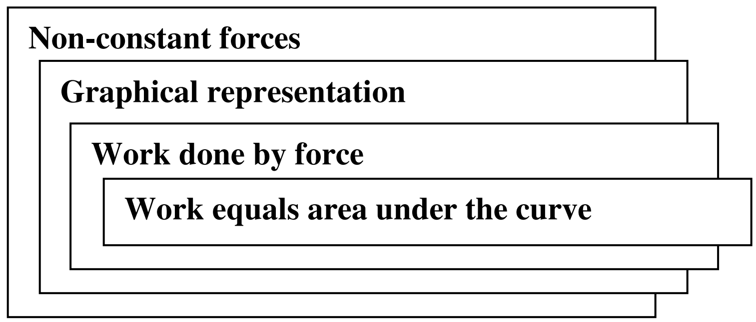
Interviewer: *What does the area represent then?*

Eric: *It's work of the spring, right?*

Interviewer: *Alright, so let's do that.*

This was the first time Eric encountered a physics problem involving the area under the curve (as he said after the interview). He started out trying to find the values for k and x to plug in the formula for the work done by the spring force. Upon being hinted to exploit the graph, he thought of the slope of the graph though he was not sure what physical quantity the slope represented. Although he had learned from the lecture that the work done by a force could be obtained by finding the area under the curve of force versus displacement, he failed to recognize it in this problem. However, he was eventually able to recall that knowledge resource when being asked to think of what he had learned in the lecture that was related to the graph. He also found the method applicable to the problem at hand ("*I can find the area under the curve*"). In the language of the transfer in pieces framework, we say Eric had developed a concept projection for the area under the curve. Eric's realization of the applicability of the area under the curve concept to this problem was based solely upon a single knowledge resource: "the work equals area under the curve." He mentioned neither the accumulating of incremental works nor the condition under which the area was the value of work (i.e. the graph must be that of force versus linear displacement). The knowledge frame that guided Eric as he calculated the work done by the spring using the area under the curve in interview 2 is presented in Figure 7.10.

Figure 7.10 Eric's knowledge frame that guided his thinking in the graphical problem in interview 2



Next, we will see how Eric's concept projection for the area under the curve developed as he went through the later interviews.

7.3.2.2 Interview 3 – Graphical problem

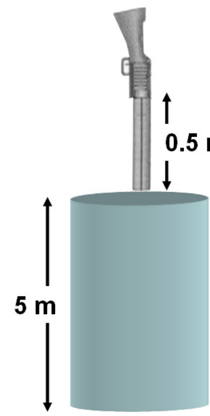
The graphical problem in this interview is presented in Figure 7.11 below. The initial spring compression could be obtained by using the work-kinetic energy theorem, in which the work done by the spring was the value of the area under the curve (i.e. the triangular section on the graph).

Eric was presented this problem after he had completed the algebraic problem in this interview. The strategy to solve this graphical problem was identical to the previous algebraic problem. The only difference was that in the algebraic problem, the work done by the resistance force was calculated by computing the integral $\int F(x)\,dx$, while in the graphical problem, the work was calculated by finding the area under the curve of force.

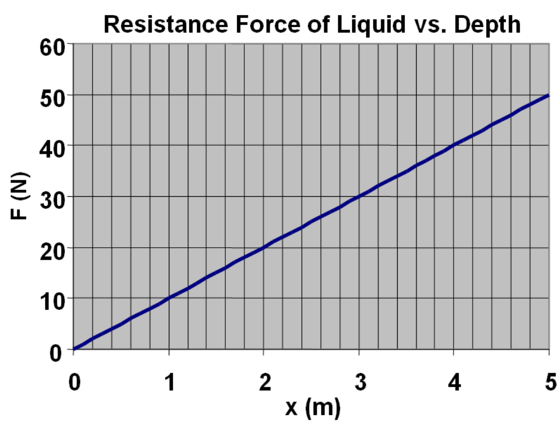
Since Eric had completed the algebraic problem, he had no difficulty setting up the equation for the work-kinetic energy theorem in the graphical problem. He also easily recognized that the work done by the resistance force was the area under the curve by relating to the integral in the algebraic problem. The following excerpt is from the beginning of the problem.

Figure 7.11 The graphical problem in interview 3

A 0.1 kg bullet is loaded into a gun compressing a spring which has spring constant k = 6000 N/m. The gun is tilted vertically downward and the bullet is fired into a drum 5.0 m deep, filled with a liquid.



The barrel of the gun is frictionless. The resistance force provided by the liquid changes with depth as shown in the graph below. The bullet comes to rest at the bottom of the drum. What is the spring compression *x* ?



Interviewer: *What are the similarities and differences between this problem and the previous one?*

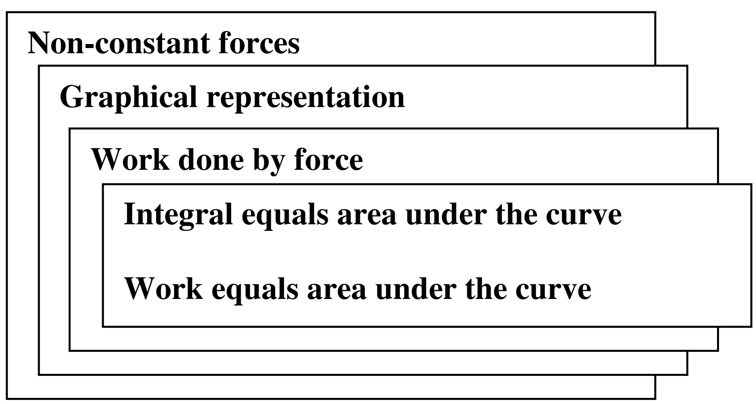
Eric: *Just that the frictional F is given in a graph instead of an equation. Everything else is the same. All I'll have to do is to find the area of the graph.*

Interviewer: *What physical quantity does the area represent?*

Eric: *It's the same thing as the integral in the previous problem. It will give me the work done by the frictional force.*

In this interview, Eric easily recognized the applicability of the area under the curve in finding the work done by the resistance force. There were two knowledge resources that Eric explicitly mentioned in his reasoning: “the integral equals the area under the curve” and “the work equals the area under the curve.” In the language of the transfer in pieces framework, we say that Eric had the concept projection of the area under the curve, which consisted of two knowledge resources compared to only one resource as in interview 2. This expansion in the knowledge base of Eric’s concept projection of the area under the curve allowed him to easily see the similarities between the graphical and the algebraic problems, and hence, see the applicability of the area under the curve concept by recognizing its relationship with the algebraic integral. Figure 7.12 below shows the knowledge frame for Eric’s concept projection for the area under the curve in interview 3.

Figure 7.12 Eric’s knowledge frame that guided his thinking in the graphical problem in interview 3

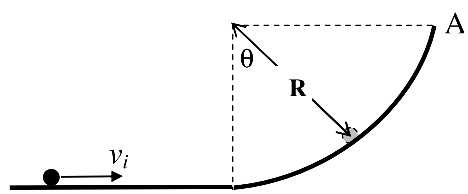


7.3.2.3 Interview 4 – Graphical problem

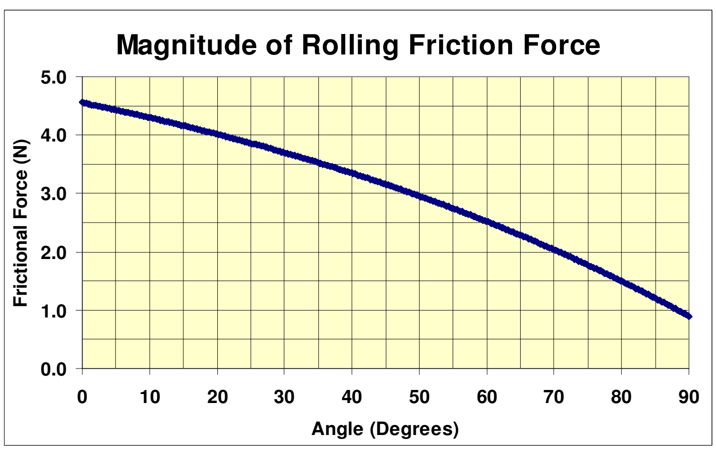
The graphical problem in this interview is presented in Figure 7.13. Similar to the graphical problems in the previous interviews, this problem could be solved using the work-kinetic energy theorem. However, the graph given in this problem was the graph of force versus angular displacement, so the area under the curve was not the value of work. To find the work done by the frictional force, students had to convert this graph to the graph of force versus linear displacement, or to multiply the area under the given curve by the radius of the track. These strategies were equivalent to calculating the integrals $\int_0^{\pi R/2} F(\theta)\,ds$ and $R\int_0^{\pi/2} F(\theta)\,d\theta$, respectively.

Figure 7.13 The graphical problem in interview 4

A sphere radius r = 1 cm and mass m = 2 kg is rolling at an initial speed v_i of 5 m/s along a track as shown. It hits a curved section (radius R = 1.0 m) and is launched vertically at point A. The rolling friction on the straight section is negligible.



The magnitude of the rolling friction force acting on the sphere varies as angle θ as per the graph shown below. What is the launch speed of the sphere as it leaves the curve at point A?



Prior to this problem, Eric had completed the algebraic problem in which the work done by the frictional force was calculated by the integral $\int_0^{\pi R/2} F(\theta)\,ds$. When he started the graphical problem, he was able to relate it with the algebraic problem but in inappropriate ways.

Interviewer: *What are the similarities and differences between this problem and the previous problem?*

Eric: *Same principles apply. All the values are the same except instead of an equation you are given a graph to find the friction. So I need to figure out how to do the integral so I'll need the area under this graph.* [calculated the area and got 267.5]

Interviewer: *What quantity does the area under this graph represent?*

Eric: *It's work, isn't it?*

Interviewer: *What's the unit of your area?*

Eric: *Shouldn't it be Joule?*

Interviewer: *You find the area, which means you multiply the quantities on the vertical and horizontal axes. Their units must be multiplied too.*

Eric: *So … Newton times degree.*

Interviewer: *Is that the unit you expect for work?*

Eric: *No, I want Joule … or Newton times meter.*

Interviewer: *So how do you convert degree to meter?*

Eric: *One revolution would be 360 degrees but I don't know where meter comes in.*

Interviewer: *What unit should the conversion factor carries then?*

Eric: *Meter over degree.*

Interviewer: *So how many meters correspond to how many degrees?*

Eric: *I don't know.*

Interviewer: *Let's consider a circle. What are the angles and circumference?*

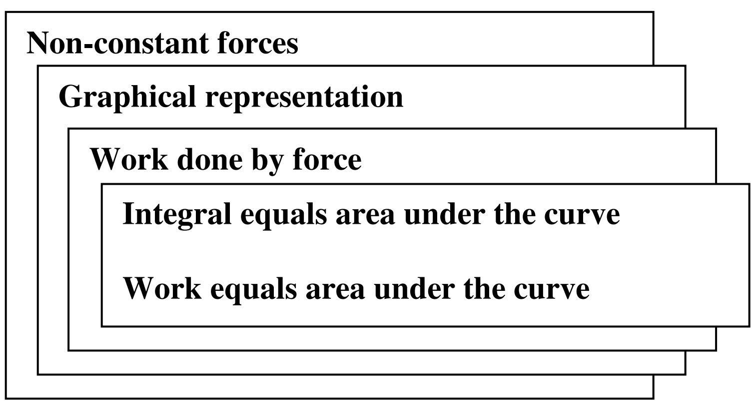
Eric: *One revolution is 360 degrees … or 2π … and circumference is $2\pi R$.*

Interviewer: *So what is the conversion factor then?*

Eric*: $2\pi R$ meters over 360 degrees.* [did the unit conversion and got the correct value of work]

At the beginning of the problem, Eric stated that the area under the curve was the value of the work done by friction by invoking two knowledge resources: the integral equaled the area under the curve ("*So I need to figure out how to do the integral so I'll need the area under this graph"*); and the work equaled the area under the curve of force ("*It's work, isn't it?"*). However, these two knowledge resources were not appropriately used in this case. First, Eric recognized the use of the area under the curve by relating it with the integral he encountered in the algebraic problem, though there was a difference between them. The integral in the algebraic problem was $\int_0^{\pi R/2} F(\theta)\, ds$ which corresponded to the area under the curve of force versus length along the curve while the graph provided was that of force versus angle. Second, the area under the curve equaled the value of work only if the curve was of force versus linear displacement, which was not the case of the graphical problem at hand. So the concept projection consisting of these two knowledge resources, which Eric successfully used in the graphical problem in interviews 2 and 3, is not applicable in this problem. In the transfer in pieces language, Eric's concept projection for the area under the curve did not span the type of problems in which the graph provided was not that of force versus linear displacement. The missing of the knowledge resource that the area under the curve was the value of work *only* in the case that the curve was on a graph of force versus linear displacement in Eric's concept projection for the area under the curve led to his overuse of the concept. Figure 7.14 below shows the knowledge frame for Eric's concept projection for the area under the curve in interview 3

Figure 7.14 Eric's knowledge frame that guided his thinking in the graphical problem in interview 4



7.3.3 Summary of Case Study 2

In interview 2 through interview 4, Eric developed and applied his concept projections for the area under the curve method for calculating the work done by non-constant forces. At his first encounter with the task, Eric did not have a concept projection for the area under the curve, so he did not spontaneously recognize the applicability of the area under the curve concept in the graphical problem of interview 2. Only upon being hinted to recall the strategy for finding the work from a graph that he had learned in the lecture was Eric able to invoke the knowledge resource that “the work equaled the area under the curve of force.” This knowledge resource by itself constituted the entire knowledge base for Eric’s first concept projection for the area under the curve. This rudimental concept projection was adequate for Eric to solve the graphical problem in interview 2, because the graph provided was that of force versus linear displacement and the area under the curve was the value of work.

In interview 3, Eric added another knowledge resource – “the integral equaled the area under the curve” – to the knowledge base for his concept projection for the area under the curve by relating it with the integral in the algebraic problem. This expansion in the knowledge base of the concept projection allowed him to perceive the algebraic and graphical problems as very similar (“*All I'll have to do is to find the area of the graph”*) although they were presented in different representations. Having carried his concept projection consisting of those two knowledge resources into interview 4, Eric spontaneously recognized that finding the area under the curve was a way of doing the integral which would result in the value of work. However, the graph provided in interview 4 was that of force versus angular displacement and hence the area under the curve was not yet the value of work. The knowledge base of Eric’s concept projection contained the resource that “work is the area under the curve”, but did not contain the resource that “the curve must be on a graph of force versus linear displacement.” Thus Eric failed to recognize that the area under the curve in interview 4 was not the value of work.

We answer our research questions for this case study as follows:

- What resources did Eric use to recognize the applicability of the area under the curve concept in calculating the work?

At the first time Eric encountered the graphical problem, he was not able to activate the resource that “the work equaled the area under the curve of force.” As he progressed through the interviews, he was able to activate that resource, as well as the resource that “the integral equaled the area under the curve.” However, he was not able to recognize that “the curve must be on a graph of force versus linear displacement” so he failed to recognize that the area under the curve of force in interview 4 was not yet the value of work.

- How did Eric's concept projection of the integral concept change as he progressed through the interviews?

The first time Eric encountered the graphical problem, he did not have a concept projection for the area under the curve concept in calculating the work. Upon being prompted, Eric attended to the fact that he was provided a graph and needed to calculate the work, which cued the activation of the resource that "the work equaled the area under the curve of force." In interview 3, Eric did the algebraic problem before the graphical problem so he also attended to the integral-area relation and used it as a cue for finding the work using the area under the curve concept. This was a development in his concept projection of the area under the curve concept which allowed him to perceive the graphical and the algebraic problems as similar. However, Eric did not attend to the variable on the graph of force provided, so he did not recognize that the area under the given curve was not the value of work. In other words, Eric's concept projection did not span the type of problems in which the force was provided as a graph of force versus a variable other than linear displacement.

7.4 Conclusions

We have utilized Wagner's theoretical framework of concept projection to interpret and trace the development of individual students' application of the integral and the area under the curve concepts to physics problems. We found that at the first time the students encountered the task, they usually did not have a concept projection for the integral and the area under the curve concepts. Hence, they did not recognize the use of these concepts in the problems at hand. Instead, they relied on pre-determined formulae ($W = F \cdot d$ in the case of Alex and $W = \frac{1}{2}kx^2$ in the case of Eric) to calculate the desired quantities. With the hints provided by the interviewer, these students were able to recall one knowledge resource they had learned in the course, which constituted the first, rudimental knowledge base for their concept projections for the concepts. Activating this resource helped students recognize the use of those concepts in the problems at hand. As students progressed through the interviews, they gradually enriched the knowledge bases for their concept projections by adding more knowledge resources, which allowed them to perceive the concepts as applicable to a broader range of problems. In other words, students expanded the span of their concept projections as they progressed through the interviews.

This finding aligns with Wagner's transfer in pieces framework as per which "transfer is understood … as the incremental growth, systematization, and organization of knowledge resources that only gradually extend the span of situations in which a concept is perceived as applicable." (Wagner, 2006, p. 10). However, we also found from our studies the instances when students' concept projections did not span the problems asked in our interview. In such situations, students either did not perceive the concept as applicable in the problem when it was in fact applicable, or conversely they perceived it as being applicable when it was in fact not applicable. We found the latter behavior in our studies. Alex claimed the integral of the force function $\int_0^{\pi/2} F(\theta)\,d\theta$ as the work done by the force while it was not. In this case, the missing of one important knowledge resource, Alex's concept projection for the integral – "the integral must be that of force versus linear displacement $\int Fds$" – led to his overuse of the integral concept. Similarly, Eric claimed that the area under the curve of $F(\theta)$ vs. θ was the work done by the force when, in fact it was not so. The missing of an important knowledge resource Eric's concept projection for the area under the curve – "the curve must be that of force versus linear displacement, i.e. $F(x)$ vs. x" – led to his overuse of the area under the curve concept.

7.5 Implications for instruction

The two case studies presented in this chapter revealed the resources, readout and reasoning strategies that students used to determine the applicability of the integral and the area under the curve concepts in calculating the work done by non-constant forces in work-energy problems. Students' success and failure on integration tasks were also explained based on the resources activated, the readout and reasoning strategies used. These studies suggested that students' success or failure in recognizing the applicability of a concept depended upon the resources and the cognitive strategies the students' used. When a student did not activate a certain resource that was necessary in the situation, it was possible that he did not have that resource (lack of a resource) or he was unaware of the type of information given (lack of a readout strategy). The case studies presented in this chapter suggested a strategy for facilitating students' problem solving. The strategy required physics instructors to be aware of the resources students are possessing and/or missing, and be prepared to provide appropriate prompting that might cue students to activate the necessary resources and prompting that guides students attention to certain features of the situation at hand.

In our tutorials, the sequence of math and physics exercises in tutorial 3 prepared students with the appropriate resources necessary for work-energy problems involving non-constant forces. The math and physics exercises in tutorial 4 prepared the students with the resources and also prompted students' attention to the variable of the given force function. The promising results of these tutorials in helping students learn to apply the integral and the area under the curve concepts in work-energy problems indicated that the strategy we proposed above might be an effective strategy to facilitate students' problem solving.

Chapter 8 - Summary and Conclusions

8.1 Summary of this research project

The research project presented in this dissertation has two phases. In phase 1 of the project, we investigated the difficulties students encountered in applying the integral and the area under the curve concepts in physics problem solving, and the hints that might help them overcome those difficulties. Based on the findings of phase 1, in phase 2 of the project, we created tutorials to facilitate students' learning to solve physics problems with integral and the area under the curve, and tested their effectiveness in comparison to typical textbook problems. We summarize the results of each phase of the project below.

8.1.1 Results from phase I of the project

The application of the integral and the area under the curve concepts in physics problem solving can be broken up into four steps:

- Step 1: recognize the need for an integral
- Step 2: set up the expression for the infinitesimal quantity
- Step 3: accumulate the infinitesimal quantities
- Step 4: compute the integral

We investigated students' difficulties with each of these steps in the context of mechanics and electricity. In mechanics, our interview problems involved calculating the work done by a non-constant force from the force function. We found that not many students were able to recognize the relation $W = \int F(x)\,dx$ although they had learned about it in the course. Students' unfamiliarity with the task might explain for this result. We also found that most of the students did not think about the integral as an accumulation of small quantities to obtain the total quantity, so they had difficulties setting up the correct integral for the work done by a force when the force was given as a function of angular displacement.

Our interview problems in electricity involved calculating several physical quantities (e.g. electric field, resistance, capacitance, electric current) from other non-constant quantities (e.g. charge distribution, resistivity, current density). We found that students did not have significant difficulties recognizing the need for an integral in a physics problem, although there were still a few students who did not recognize it. The non-constant nature of a physical quantity provided in the problem statement was found to be the major cue for students to think of using integration in the problem. We found the real stumbling block to be the step 2: setting up the expression for the infinitesimal quantity. Most of the students in our study did not indicate an understanding of the accumulation process underlying the integral. Students did not think of the total quantity as a sum of infinitesimal quantities, even when being prompted by the interviewer. Most of them were unable to set up the correct expression for the infinitesimal quantity because they did not understand what the "infinitesimal quantity" meant, and could not interpret the meaning of the infinitesimal term (i.e. dx, dr, …) in the integral. Once the expression for the infinitesimal quantities was set up, students usually integrated it immediately without noticing how the quantities should be added up. This tendency led students to errors when the quantities to be accumulated were vector quantities (e.g. electric field) or quantities that must be added reciprocally (e.g. capacitance of a series of capacitors). We also found that students had difficulties computing the integrals that were set up. These difficulties included determining variables and constants in an integral, determining the limits of the integral, converting one variable to another.

Most of the students' difficulties described above were due to students not understanding the accumulation process underlying the integral. So the hints that helped them overcome those difficulties guided them to think about the integral as a sum of infinitesimal quantities. This could be done by analyzing the structure of the integrand and interpreting the meaning of each of the terms and symbols in the integral.

We also investigated students' application of the area under the curve concept in physics problems in mechanics and electricity. In these problems, the integrals must be evaluated graphically using the area under the curve concept. In mechanics, the problems involved calculating the work done by a non-constant force from the graph of force versus displacement. Only a few students could recognize that the work equaled the area under the curve of $F(x)$ vs. x, although they had learned about it. Students failed to recognize that the area under the curve of $F(\theta)$ vs. θ was not the value of work. This error indicated that students might not know what quantity was being accumulated when calculating the area under the curve. Asking students about the structure of the Riemann sum underlying the calculation of the area under the curve helped students understand the physical meaning of the area under the curve.

In electricity problems, students had to compute the integrals set up from the problem statements by calculating the area under the curve when there were several graphs provided. We found that many students had significant difficulties relating an integral with the corresponding are under the curve. Asking students to label a graph knowing that the area under the curve in that graph equaled a certain integral was the strategy that helped students recognize the correct graph to find the area.

8.1.2 Answers to the research questions in phase I of the project

8.1.2.1 Students' application of the integral concept in mechanics

RQ1: To what extent did students recognize the use of the integral in physics problems?

Most of the students were not able to recognize the use of the integral in calculating the work done by non-constant forces. Instead, they attempted to use pre-derived formulas to calculate the work. Students' inability to recognize the use of the integral might be attributed to their unfamiliarity with the task (since students did not have any problems involving integral prior to our interviews) and their strong inclination to using the pre-derived formulas rather than attempting an unfamiliar strategy or inventing a new strategy.

RQ2: To what extent did students understand what quantity was being accumulated when calculating an integral?

The fact that some students knew that they had to calculate the derivative or the integral of force but did not know which one suggested that these students did not understand the physical meaning of the operators. Therefore, students' application of the integral in finding work might simply be the recall of the previously learned knowledge (i.e. the work equaled the integral) rather than an understanding of how the work was being accumulated.

The fact that most of the students claimed the integral $\int F(\theta)\,d\theta$ in interview 4 as the value of force indicated that these students did not understand what quantity was being accumulated when they performed the integral.

RQ4: What verbal hints may help students overcome those difficulties?

For students who attempted to use pre-derived formulas they learned from the course to calculate the work, the hints were to help them recognize that those formulas were not applicable to the problems at hand. For example, when a student attempted to find the spring constant using $k = \frac{F}{x}$ to plug in the formula $W = \frac{1}{2}kx^2$, the hint was to ask them whether the spring constant was actually a constant, which helped them recognize that the concept of "spring constant" did not apply for non-linear spring and hence the formula $W = \frac{1}{2}kx^2$ did not apply either. The hints that guided students to think of the non-constant nature of the force triggered students' thinking of integration. The hints on the accumulation of the infinitesimal work to get the total work also helped some students to set up the correct integral for the work in interview 4, although the hints on units seemed to be easier to understand for the students.

8.1.2.2 Students' application of the integral concept in electricity

RQ3: What are the common difficulties that students encounter when solving problems in electricity involving integration?

Students generally did not have significant difficulty recognizing the need for integration in a problem. However, students did have significant difficulties setting up and computing the desired integral. These difficulties included setting up an incorrect expression for the infinitesimal quantity and/or accumulating the infinitesimal quantities in an inappropriate manner. Determining the limits of the integrals, relating variables in an integral, and computing the integrals algebraically were also the difficulties faced by some of the students.

8.1.2.3 Students' application of the area under the curve concept

RQ1: To what extent did students recognize the use of area under the curve in physics problems?

The majority of students in our interviews did not spontaneously recognize the use of area under the curve in calculating work from the graph of force. There were two possible explanations: (i) students were not familiar with the method; and (ii) students held strong preference on algebraic method. The fact that more students were able to recognize that work equaled the area under the curve as they progressed through the interviews suggested that students gained familiarity with the concept. Some students, while talking to the interviewer after the interviews, stated that they had not seen any problem using the area under the curve in their physics homework or exam. On the other hand, students also expressed an inclination to an algebraic approach even when a graph was provided. They attempted to use pre-derived formulae for work and just used the graph to collect data on the values of spring constant or coefficient of friction to plug in those formulae. Some students explicitly told the interviewer that they hated problems with graphs and preferred working with equations. These facts supported the second explanation.

RQ2: To what extent did students understand what quantity was being accumulated when calculating the area under a curve?

In the graphical problems in interviews 2 and 3, the area under the curve itself was the value of work. So when a student recognized that work equaled the area under the curve, we did not know whether he understood how work was accumulated when calculating the area or he just applied what he was taught in the lecture. There were four students in interview 2 stated that the area had some meaning but were not able to tell what the meaning was, and three students in interview 3 stated that the slope of the line was the coefficient of friction. These were evidence that these students did not understand what quantity the slope and the area represented.

In the graphical problem in interview 4, finding the area meant accumulating the product of force and angle, which did not yield the total work. Six out of 9 students spontaneously stated that work equaled the area under the curve, but only one of them recognized the need for the radius factor without assistance from the interviewer. This was further evidence that although students could invoke the knowledge of "work equaled the area under the curve of force," they might not understand what quantity was being accumulated when calculating such an area. Therefore, they failed to apply that knowledge in novel situations.

RQ3: To what extent did students understand the relationship between a definite integral and area under a curve?

Almost all of the students indicated knowledge of "the integral equaled the area under the curve," but only half of them (four students in interview 5, eight in interview 6, and nine in interview 7) were able to select the graph corresponding to a pre-determined integral when several graphs were present. The errors other students made – choosing a graph based on part of the integrand or on the simplicity of the area calculation – indicated that these students did not completely understand the relationship between a definite integral and area under a curve.

8.1.3 Results from phase II of the project

In phase 2 of the project, we created tutorials to facilitate students' learning to solve physics problems involving integration in mechanics an electricity. In mechanics, the tutorials aimed at helping students solve problems on work-energy, in which the work done by a force must be calculated by the integral or the area under the curve of force. The tutorial consisted of a sequence of math and physics exercises, a debate problem, and two problem posing tasks. The sequence of math and physics exercises provided students with the opportunity to activate a mathematical model or knowledge in a context-free math exercise and then apply it to a simple physics context. The debate problem prepared students with the physics background necessary to solve problems on the topic. The problem posing tasks were intended to help students practice putting together a mathematical model in a physics context. We compared the effectiveness of our tutorials in comparison with textbook-style problems (which we called "standard materials"). In this dissertation we discussed two tutorials on the topic of work-energy. We found that for both tutorials, students in the treatment group learning with our tutorial materials outperformed students in the control group learning with standard materials on integral related tasks, although there was no difference between the two groups on physics related tasks. This result suggested that out tutorials helped students learn about integration better than standard material did, but post-test scores around 50% indicate that there is still room for improvement. The tutorials still needed to be enhanced to better prepare students with the physics background of the problems.

The tutorials in electricity employed a different strategy. For most problems in introductory electricity, setting up the integral describing the physical quantities was the major part of the solution, and the integrals were more complicated than those in mechanics. For these reason, the tutorials in electricity aimed at helping students set up the integrals by braking up the process in smaller steps so that students could learn how complicated integrals were formed by translating the physical situation described in the statement to mathematical notations. Each of the tutorials in electricity consisted of a physics problem in which a physical quantity was calculated using the integral. This problem was broken up into several steps or smaller exercises, through which students were led from the simple case with discrete quantities to more and more complicated cases where there were several quantities or infinites number of infinitesimal quantities. We also tested our tutorials in comparison to textbook-type problems (standard materials). We found that two out of the three tutorials that we created in this study helped students learn to set up an integral in a physics problem more than the standard material did, but the third tutorial did not make a significant improvement. The reason for this might be that the students were unfamiliar with the type of integral in the third tutorial. Analysis of the types of errors students made in this tutorial implementation supported this hypothesis. Although the students in the treatment group learning with our tutorials outperformed the students in the control group learning with standard materials, there were still fewer than half of the students who could solve the problems in the post-test. This implied that our tutorials needed to be improved to help a larger proportion of students learn about integral in electricity problems.

8.1.4 Answers to the research questions in phase II of the project

8.1.4.1 Tutorials in mechanics

To what extent did our tutorials help students improve their ability to apply the integral and the area under the curve concepts in work – energy problems, compared to standard instruction (i.e. sample problems and solutions)?

Both of our tutorials on the topics of work – energy for a point mass and for a rigid body significantly improved students' ability to calculate a physical quantity using the integral and the area under the curve concepts in a physics problem, although they were not so effective in preparing students with the physics background of the work – energy problems. These results suggested that the tutorials should be improved to better prepare students with the physics background of the problems.

8.1.4.2 Tutorials in electricity

To what extent did our tutorials help students improve their ability to apply the integral concept to electricity problems, in comparison to standard materials (i.e. sample problems and solutions)?

The first two tutorials on the electric field and the resistance problems improved students' ability to apply the integral into physics problems significantly more than the standard materials did. However, the percentage of students being able to obtain the correct answer in the post-test in these two FOGLI (Focus Group Learning Interview) sessions was still less than 50%. So our tutorials must be improved to facilitate more students to learn about integration in electricity problems. The third tutorial did not provide promising results as the other two. The major reason for this might be that the students were unfamiliar with the type integral in the third tutorial. This tutorial needed to be revised to teach students even more about integrating with respect to area.

8.2 What's new in my research?

There have been many studies in physics education research on students' problem solving with multiple representations (e.g. numerical, algebraic, graphical, tabular). Most of these studies focus on which representations students choose to use, how they use the representation when solving physics problems, and the correlation between the representation students use and their success in solving the problems. My research also involves several representations (e.g. numerical, algebraic, graphical), but the focus is on how students extract information and calculate physical quantities from the representations.

There have been many studies on how students use integration in physics problems. However, these studies mostly discuss the first step (recognizing the need for an integral) and the last step (computing an integral, e.g. confusing between variables and constants, incorrect limits,). The new idea in my study is that I break up the application of the integral into four steps and investigate students' difficulties with each of the steps. Therefore, my study provides a closer look and more detailed insights into student's difficulties when applying the integral to physics problems.

There have been a few studies on students' interpretation of graphs in kinematics (e.g. McDermott, 1986) and thermodynamics (e.g. Pollock, 2007). My study investigates how students use graphs in many other topics of physics, including work done by a force and electricity. The new idea in my study is that I also investigate how students use graphs to evaluate definite integrals by providing the students with several graphs instead of just the graph related to the integral. This helps reveal students' understanding of the integral-area relation.

There have been many tutorials created to help students learn physics (e.g. Tutorials in Introductory Physics, Activity-Based Tutorials, Open-Source Tutorials). These tutorials aim at improving students' conceptual understanding in physics. However, there are no tutorials focusing on helping students learn to apply mathematical concepts in solving physics problems.

In my study, I have created tutorials that aim at helping students' learn to apply one of the most important mathematical concepts – the integral – in physics problems.

8.3 Implications for instruction

My research has revealed the common difficulties students encountered in applying the integral and the area under the curve concepts in physics problems. Based on these findings, we created tutorials to facilitate students' application of these concepts in physics problem solving. These works have many implication for both mathematics and physics instruction, and open new directions for future research.

All of the participants in our research were students in the Engineering Physics course sequence at Kansas State University. At least one and two semesters of calculus were required for enrolling in the Engineering Physics 1 and 2 courses, respectively. This means that students have had quite intensive training in calculus before participating in our research. However, students' performance on integration related tasks in our research indicated that such training in calculus did not prepare them well enough to apply their calculus knowledge to physics problems solving. Our research found that students did not think of integration as an accumulation process and did not understand the meaning of the integrand and the infinitesimal term in the integral, therefore they had difficulties setting up the integral from the physics situations. So we suggest that instruction in calculus should focus more on the accumulation process underlying integration. This could be done by providing the students with problems which ask students to compute the integrals using the Riemann sum method rather than using pre-determined integral techniques. Calculus homework and examination should include more application problems to provide students with the opportunities to apply the calculus concept to physics problems right after students have learned about it in calculus. This is similar to the sequence of math and physics exercises in our tutorials, which have been proven to improve students' application of integration in physics problems.

As discussed above, students entering calculus-based physics courses might not have satisfactory understanding of calculus concepts and their calculus knowledge might not be ready to be applied to physics. Therefore, physics instruction should include tutorials on calculus concepts to provide students the opportunity to enhance their calculus knowledge and more importantly, to learn how calculus concepts are applied to solve physics problems. The tutorials we created in this research give an example of the kind of mathematics tutorials that have been proven to be helpful. Due to time limitation, our tutorials were pretty short and therefore could only focus on particular types of physics problems. Physics instructors should create more tutorials covering a broader range of topics and problem types, and implement those tutorials as a regular activity during recitation or problem solving sessions in the course.

8.4 Possibilities for further research

One of the limitations of the research presented in this dissertation is the small number of students participating in the studies of phase I of the project. There were only 20 participants in comparison to more than 200 students enrolled in the course. So, further research with larger sample sizes is needed to verify the results of these studies. Our study has suggested a four-step model for investigating students' application of the integral in physics problems. This model might serve as a lens for other researchers to look at students' performance on integration tasks in physics.

The tutorials we created in this project are limited in quantity and topics (two tutorials on work-energy and three tutorials on electricity), in the types of problems, and in the amount of learning experience they provide. However, the results from implementing these tutorials are promising. These results encourage researchers to develop more tutorials that cover a broader range of topics and problem types, and also increase the amount of learning experience students might have from using the tutorials.

Integration is a dynamic process, in the sense that a large object is chopped into infinitesimal pieces on which the physical quantity is evaluated and then is accumulated over all pieces to obtain the total quantity. Therefore, computer simulation might be employed to demonstrate the chopping and accumulating process happening when an integral is performed. Developing computer simulations on the application of calculus concepts in physics is a promising direction for future research.

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