Quantization of the Free Electromagnetic Field: Photons and Operators

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Summary

The main ideas and equations for quantized free electromagnetic fields are developed and summarized here, based on the quantization procedure for coordinates (components of the vector potential $\mathbf{A}$) and their canonically conjugate momenta (components of the electric field $\mathbf{E}$). Expressions for $\mathbf{A}$, $\mathbf{E}$ and magnetic field $\mathbf{B}$ are given in terms of the creation and annihilation operators for the fields. Some ideas are proposed for the interpretation of photons at different polarizations: linear and circular. Absorption, emission and stimulated emission are also discussed.

1 Electromagnetic Fields and Quantum Mechanics

Here electromagnetic fields are considered to be quantum objects. It's an interesting subject, and the basis for consideration of interactions of particles with EM fields (light). Quantum theory for light is especially important at low light levels, where the number of light quanta (or photons) is small, and the fields cannot be considered to be continuous (opposite of the classical limit, of course!).

Here I follow the traditinal approach of quantization, which is to identify the coordinates and their conjugate momenta. Once that is done, the task is straightforward. Starting from the classical mechanics for Maxwell's equations, the fundamental coordinates and their momenta in the QM system must have a commutator defined analogous to $[x, p_x] = i\hbar$ as in any simple QM system. This gives the correct scale to the quantum fluctuations in the fields and any other dervied quantities. The creation and annihilation operators will have a unit commutator, $[a, a\dagger] = 1$, but they have to be connected to the fields correctly. I try to show how these relations work. Getting the correct normalization on everything is important when interactions of the EM fields with matter are considered.

It is shown that the quantized fields are nothing more than a system of decoupled harmonic oscillators, at a collection of different wavevectors and wave-polarizations. The knowledge of how to quantize simple harmonic oscillators greatly simplifies the EM field problem. In the end, we get to see how the basic quanta of the EM fields, which are called photons, are created and annihilated in discrete processes of emission and absorption by atoms or matter in general. I also want to discuss different aspects of the photons, such as their polarization, transition rules, and conservation laws.

Later in related notes I'll talk about how this relates to describing two other topics of interest: quantum description of the dielectric response of materials (dielectric function $\epsilon(\omega)$), and, effects involving circularly polarized light incident on a material in the presence of a DC magnetic field (Faraday effect). I want to describe especially the quantum theory for the Faraday effect, which is the rotation of the polarization of linearly polarized light when it passes through a medium in a DC magnetic field parallel to the light rays. That is connected to the dielectric function, hence the interest in these related topics.

1.1 Maxwell's equations and Lagrangian and Hamiltonian Densities

A Lagrangian density for the free EM field is

$$\mathcal{L} = \frac{1}{8\pi} \left[\mathbf{E}^2 - \mathbf{B}^2 \right] \tag{1}$$

[It's integral over time and space will give the classical **action** for a situation with electromagnetic fields.] This may not be too obvious, but I take it as an excercise for the reader in classical mechanics, because here we want to get to the quantum problem. Maxwell's equations in free space, written for the electric (**E**) and magnetic (**B**) fields in CGS units, are

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0 \tag{2}$$

The zero divegence of **B** and Faraday's Law (1st and 2nd eqns) allow the introduction of vector and scalar potentials, **A** and Φ, respectively, that give the fields,

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{3}$$

I consider a problem far from sources. If sources were present, they would appear in the last two equations in (2), so these same potentials could still apply. The potentials are not unique and have a gauge symmetry. They can be shifted using some guage transformation (f) without changing the electric and magnetic fields:

$$\mathbf{A}' = \mathbf{A} + \nabla f, \quad \Phi' = \Phi - \frac{1}{c}\frac{\partial f}{\partial t} \tag{4}$$

The Euler-Lagrange variation of the Lagrangian w.r.t the **coordinates** $q = (\Phi, A_x, A_y, A_z)$ gives back Maxwell's equations. Recall Euler-Lagrange equation and try it as a practice problem in classical mechanics.

To approach quantization, the canonical momenta p_i need to be identified. But there is no time derivative of Φ in $\mathcal{L}$, so there is no p_Φ and Φ should be eliminated as a coordinate, in some sense. There *are* time derivatives of **A**, hence their canonical momenta are found as

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \frac{1}{4\pi c}\left(\frac{\partial \Phi}{\partial x_i} + \frac{1}{c}\frac{\partial A_i}{\partial t}\right) = -\frac{1}{4\pi c}E_i, \quad i = 1, 2, 3 \tag{5}$$

The transformation to the Hamiltonian energy density is the Legendre transform,

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L} = \mathbf{p} \cdot \frac{\partial \mathbf{A}}{\partial t} - \mathcal{L} = 2\pi c^2 \mathbf{p}^2 + \frac{1}{8\pi}\left(\nabla \times A\right)^2 - c\mathbf{p} \cdot \nabla\Phi \tag{6}$$

When integrated over all space, the last term gives nothing, because $\nabla \cdot \mathbf{E} = 0$, and the first two terms give a well known result for the effective energy density, in terms of the EM fields,

$$\mathcal{H} = \frac{1}{8\pi}\left(\mathbf{E}^2 + \mathbf{B}^2\right) \tag{7}$$

We might keep the actual Hamiltonian in terms of the coordinates **A** and their conjugate momenta **p**, leading to the classical EM energy,

$$H = \int d^3r \left[2\pi c^2 \mathbf{p}^2 + \frac{1}{8\pi}\left(\nabla \times \mathbf{A}\right)^2\right] \tag{8}$$

Now it is usual to apply the **Coulomb gauge**, where $\Phi = 0$ and $\nabla \cdot \mathbf{A} = 0$. For one, this insures having just three coordinates and their momenta, so the mechanics is consistent. Also that is consistent with the fields needing three coordinates because they are three-dimensional fields. (We don't need six coordinates, because **E** and **B** are not independent. In a vague sense, the magnetic and electric fields have some mutual conjugate relationship.) We can use either the Lagrangian or Hamiltonian equations of motion to see the dynamics. For instance, by the Hamiltonian method, we have

$$\dot{q}_i = \frac{\delta \mathcal{H}}{\delta p_i}, \quad \dot{p}_i = -\frac{\delta \mathcal{H}}{\delta q_i} \tag{9}$$

Recall that the variation of a density like this means

$$\frac{\delta \mathcal{H}}{\delta f} \equiv \frac{\partial \mathcal{H}}{\partial f} - \sum_i \frac{\partial}{\partial x_i}\frac{\partial \mathcal{H}}{\partial\left(\frac{\partial f}{\partial x_i}\right)} - \frac{\partial}{\partial t}\frac{\partial \mathcal{H}}{\partial\left(\frac{\partial f}{\partial t}\right)} \tag{10}$$

The variation for example w.r.t coordinate $q_i = A_x$ gives the results

$$\frac{\partial A_x}{\partial t} = 4\pi c^2 p_x, \qquad \frac{\partial p_x}{\partial t} = \frac{1}{4\pi}\nabla^2 A_x \qquad (11)$$

By combining these, we see that all the components of the vector potential (and the conjugate momentum, which is proportional to $\mathbf{E}$) satisfy a wave equation, as could be expected!

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \qquad (12)$$

Wave motion is essentially oscillatory, hence the strong connection of this problem to the harmonic oscillator solutions.

The above wave equation has plane wave solutions $e^{i\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t}$ at angular frequency $\omega_{\mathbf{k}}$ and wave vector $\mathbf{k}$ that have a linear dispersion relation, $\omega_{\mathbf{k}} = ck$. For the total field in some volume V, we can try a Fourier expansion over a collection of these modes, supposing periodic boundary conditions.

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\mathbf{A}_{\mathbf{k}}(t)\ e^{i\mathbf{k}\cdot\mathbf{r}} \qquad (13)$$

Each coefficient $\mathbf{A}_{\mathbf{k}}(t)$ is an amplitude for a wave at the stated wave vector. The different modes are orthogonal (or independent), due to the normalization condition

$$\int d^3\mathbf{r}\ e^{i\mathbf{k}\cdot\mathbf{r}}\ e^{-i\mathbf{k}'\cdot\mathbf{r}} = V\delta_{\mathbf{k}\mathbf{k}'} \qquad (14)$$

The gauge assumption $\nabla \cdot \mathbf{k} = 0$ then is the same as $\mathbf{k} \cdot \mathbf{A}_{\mathbf{k}} = 0$, which shows that the waves are transverse. For any $\mathbf{k}$, there are two transverse directions, and hence, two independent polarizations directions, identified by unit vectors $\mathbf{e}_{\mathbf{k}\alpha}$, $\alpha = 1, 2$. Thus the total amplitude looks like

$$\mathbf{A}_{\mathbf{k}} = \hat{\epsilon}_{\mathbf{k}1}A_{\mathbf{k}1} + \hat{\epsilon}_{\mathbf{k}2}A_{\mathbf{k}2} = \sum_{\alpha}\hat{\epsilon}_{\mathbf{k}\alpha}A_{\mathbf{k}\alpha} \qquad (15)$$

Yet, from the wave equation, both polarizations are found to oscillate identically, except perhaps not in phase,

$$\mathbf{A}_{\mathbf{k}}(t) = \mathbf{A}_{\mathbf{k}}\ e^{-i\omega_{\mathbf{k}}t} \qquad (16)$$

Now the amplitudes $\mathbf{A}_{\mathbf{k}}$ are generally complex, whereas, we want to have the actual field being quantized to be real. This can be accomplished by combining these waves appropriately with their complex conjugates. For example, the simple waves $A = \cos kx = (e^{ikx} + e^{-ikx})/2$ and $A = \sin kx = (e^{ikx} - e^{-ikx})/2i$ are sums of "positive" and "negative" wavevectors with particular amplitudes. Try to write $\mathbf{A}$ in (13) in a different way that exhibits the positive and negative wavevectors together,

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{2\sqrt{V}}\sum_{\mathbf{k}}\left[\mathbf{A}_{\mathbf{k}}(t)\ e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{A}_{-\mathbf{k}}(t)\ e^{-i\mathbf{k}\cdot\mathbf{r}}\right] \qquad (17)$$

[The sum over $\mathbf{k}$ here includes wave vectors in all directions. Then both $\mathbf{k}$ and $-\mathbf{k}$ are included twice. It is divided by 2 to avoid double counting.] In order for this to be real, a little consideration shows that the 2nd term must be the c.c. of the first term,

$$\mathbf{A}_{-\mathbf{k}} = \mathbf{A}_{\mathbf{k}}^{*} \qquad (18)$$

A wave needs to identified by both $\mathbf{A}_{\mathbf{k}}$ and its complex conjugate (or equivalently, two real constants). So the vector potential is written in Fourier space as

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{2\sqrt{V}}\sum_{\mathbf{k}}\left[\mathbf{A}_{\mathbf{k}}(t)\ e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{A}_{\mathbf{k}}^{*}(t)\ e^{-i\mathbf{k}\cdot\mathbf{r}}\right] \qquad (19)$$

Note that the c.c. reverses the sign on the frequency also, so the first term oscillates at positive frequency and the second at negative frequency. But curiously, both together give a real wave

traveling along the direction of $\mathbf{k}$. Based on this expression, the fields are easily determined by applying (3), with $\nabla \to \pm i\mathbf{k}$,

$$\mathbf{E}(\mathbf{r},t) = \frac{i}{2c\sqrt{V}} \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left[\mathbf{A_k}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}} - \mathbf{A_k}^*(t)\, e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{20}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{i}{2\sqrt{V}} \sum_{\mathbf{k}} \mathbf{k} \times \left[\mathbf{A_k}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}} - \mathbf{A_k}^*(t)\, e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{21}$$

Now look at the total energy, i.e., evaluate the Hamiltonian. It should be easy because of the orthogonality of the plane waves, assumed normalized in a box of volume V. We also know that $\mathbf{k}$ is perpendicular to $\mathbf{A}$ (transverse waves!) which simplifies the magnetic energy. Still, some care is needed in squaring the fields and integrating. There are direct terms (btwn $\mathbf{k}$ and itself) and indirect terms (btwn $\mathbf{k}$ and $-\mathbf{k}$).

$$\int d^3\mathbf{r}\, |\mathbf{E}|^2 = \frac{1}{4c^2 V} \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \omega_{\mathbf{k}} \omega_{\mathbf{k}'} \int d^3\mathbf{r}\, \left[\mathbf{A_k} e^{i\mathbf{k}\cdot\mathbf{r}} - \mathbf{A_k^*} e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \left[\mathbf{A_{k'}^*} e^{-i\mathbf{k}'\cdot\mathbf{r}} - \mathbf{A_{k'}} e^{i\mathbf{k}'\cdot\mathbf{r}} \right] \tag{22}$$

Upon integration over the volume, the orthogonality relation will give 2 terms with $\mathbf{k}' = \mathbf{k}$ and 2 terms with $\mathbf{k}' = -\mathbf{k}$, for 4 equivalent terms in all. The same happens for the calculation of the magnetic energy. Also one can't forget that $\mathbf{A_k}$ is the same as $\mathbf{A^*_{-k}}$. These become

$$\frac{1}{8\pi} \int d^3\mathbf{r}\, |\mathbf{E}|^2 = \frac{1}{8\pi} \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}^2}{c^2} |\mathbf{A_k}(t)|^2 \tag{23}$$

$$\frac{1}{8\pi} \int d^3\mathbf{r}\, |\mathbf{B}|^2 = \frac{1}{8\pi} \sum_{\mathbf{k}} \mathbf{k}^2 |\mathbf{A_k}(t)|^2 \tag{24}$$

Of course, $\mathbf{k}^2$ in the expression for magnetic energy is the same as $\omega_{\mathbf{k}}^2/c^2$ in that for electric energy. Then it is obvious that the magnetic and electric energies are equal. The net total energy is simple,

$$H = \frac{1}{8\pi} \int d^3\mathbf{r}\, \left(|\mathbf{E}|^2 + |\mathbf{B}|^2 \right) = \frac{1}{8\pi} \sum_{\mathbf{k}} k^2 |\mathbf{A_k}|^2 = \frac{1}{4\pi} \sum_{\mathbf{k}\alpha} k^2 |A_{\mathbf{k}\alpha}|^2 \tag{25}$$

The last form recalls that each wave vector is associated with two independent polarizations. They are orthogonal, so there are no cross terms between them from squaring.

The Hamiltonian shows that the modes don't interfere with each other, imagine how it is possible that EM fields in vacuum can be completely linear! But this is good because now we just need to quantize the modes as if they are a collection of independent harmonic oscillators. To do that, need to transform the expression into the language of the coordinates and conjugate momenta. It would be good to see H expressed through squared coordinate (potential energy term) and squared momentem (kinetic energy term).

The electric field is proportional to the canonical momentum, $\mathbf{E} = -4\pi c\mathbf{p}$. So really, the electric field energy term already looks like a sum of squared momenta. Similarly, the magnetic field is determined by the curl of the vector potential, which is the basic coordinate here. So we have some relations,

$$\mathbf{p}(\mathbf{r},t) = -\frac{1}{4\pi c}\mathbf{E} = \frac{-i}{8\pi c^2 \sqrt{V}} \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left[\mathbf{A_k}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}} - \mathbf{A_k}^*(t)\, e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{26}$$

This suggests the introduction of the momenta at each wavevector, i.e., analogous with the Fourier expansion for the vector potential (i.e., the generalized coordinates),

$$\mathbf{p}(\mathbf{r},t) = \frac{1}{2\sqrt{V}} \sum_{\mathbf{k}} \left[\mathbf{p_k}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{p_k^*}(t)\, e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{27}$$

Then we can make the important identifications,

$$\mathbf{p_k}(t) = \frac{-i\omega_{\mathbf{k}}}{4\pi c^2} \mathbf{A_k}(t) \tag{28}$$

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Even more simply, just write the electric field (and its energy) in terms of the momenta now.

$$\mathbf{E}(\mathbf{r},t) = -4\pi c \mathbf{p}(\mathbf{r},t) = \frac{-2\pi c}{\sqrt{V}} \sum_{\mathbf{k}} \left[\mathbf{p_k}(t)e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{p_k^*}(t)e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{29}$$

When squared, the electric energy involves four equivalent terms, and there results

$$\frac{1}{8\pi} \int d^3\mathbf{r} \, |\mathbf{E}|^2 = \frac{16\pi^2 c^2}{8\pi} \sum_{\mathbf{k}} \mathbf{p_k} \cdot \mathbf{p_k^*} = 2\pi c^2 \sum_{\mathbf{k}} \mathbf{p_k} \cdot \mathbf{p_k^*} \tag{30}$$

Also rewrite the magnetic energy. The generalized coordinates are the components of $\mathbf{A}$, i.e., let's write

$$\mathbf{q_k} = \mathbf{A_k} \tag{31}$$

Consider the magnetic field written this way,

$$\mathbf{B}(\mathbf{r},t) = \frac{i}{2\sqrt{V}} \sum_{\mathbf{k}} \mathbf{k} \times \left[\mathbf{q_k}(t) \, e^{i\mathbf{k}\cdot\mathbf{r}} - \mathbf{q_k^*}(t) \, e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{32}$$

and the associated energy is written,

$$\frac{1}{8\pi} \int d^3\mathbf{r} \, |\mathbf{B}|^2 = \frac{4}{4 \times 8\pi} \sum_{\mathbf{k}} \mathbf{k}^2 \mathbf{q_k} \cdot \mathbf{q_k^*} = \frac{1}{8\pi c^2} \sum_{\mathbf{k}} \omega_\mathbf{k}^2 \mathbf{q_k} \cdot \mathbf{q_k^*} \tag{33}$$

This gives the total canonical Hamiltonian, expressed in the Fourier modes, as

$$H = 2\pi c^2 \sum_{\mathbf{k}} \mathbf{p_k} \cdot \mathbf{p_k^*} + \frac{1}{8\pi c^2} \sum_{\mathbf{k}} \omega_\mathbf{k}^2 \mathbf{q_k} \cdot \mathbf{q_k^*} \tag{34}$$

Check that it works for the classical problem. To apply correctly, one has to keep in mind that at each mode $\mathbf{k}$, there are the *two amplitudes*, $\mathbf{q_k}$ and $\mathbf{q_k^*}$. In addition, it is important to remember that the sum goes over all positive and negative $\mathbf{k}$, and that $\mathbf{q_{-k}}$ is the same as $\mathbf{q_k^*}$.

Curiously, look what happens if you think that the Hamiltonian has only real coordinates, and write incorrectly

$$H_{\mathrm{oo}} = \sum_{\mathbf{k}} \left[2\pi c^2 \mathbf{p_k^2} + \frac{\omega_\mathbf{k}^2}{8\pi c^2} \mathbf{q_k^2} \right] \tag{35}$$

The Hamilton equations of motion become

$$\dot{\mathbf{p}}_\mathbf{k} = \frac{\delta H_{\mathrm{oo}}}{\delta \mathbf{q_k}} = \frac{\omega_\mathbf{k}^2}{4\pi c^2} \, \mathbf{q_k}, \qquad \dot{\mathbf{q}}_\mathbf{k} = -\frac{\delta H_{\mathrm{oo}}}{\delta \mathbf{p_k}} = -4\pi c^2 \, \mathbf{p_k} \tag{36}$$

Combining these actually leads to the correct frequency of oscillation, but only by luck!

$$\ddot{\mathbf{p}}_\mathbf{k} = \frac{\omega_\mathbf{k}^2}{4\pi c^2} \, \dot{\mathbf{q}}_\mathbf{k} = -\omega_\mathbf{k}^2 \, \mathbf{p_k}, \qquad \ddot{\mathbf{q}}_\mathbf{k} = -4\pi c^2 \, \dot{\mathbf{p}}_\mathbf{k} = -\omega_\mathbf{k}^2 \, \mathbf{q_k} \tag{37}$$

These are oscillating at frequency $\omega_\mathbf{k}$.

Now do the math (more) correctly. Variation of Hamiltonian (34) w.r.t. $\mathbf{q_k}$ and $\mathbf{q_k^*}$ are different things. On the other hand, $\mathbf{q_k}$ and $\mathbf{q_{-k}^*}$ are the same, so don't forget to account for that. It means that a term at negative wave vector is just like the one at positive wave vector: $\mathbf{q_{-k}q_{-k}^*} = \mathbf{q_k^*q_k}$. This doubles the interaction. The variations found are

$$\dot{\mathbf{p}}_\mathbf{k} = \frac{\delta H}{\delta \mathbf{q_k}} = \frac{\omega_\mathbf{k}^2}{4\pi c^2} \, \mathbf{q_k^*}, \qquad \dot{\mathbf{q}}_\mathbf{k} = -\frac{\delta H}{\delta \mathbf{p_k}} = -4\pi c^2 \, \mathbf{p_k^*} \tag{38}$$

$$\dot{\mathbf{p}}_\mathbf{k}^* = \frac{\delta H}{\delta \mathbf{q_k^*}} = \frac{\omega_\mathbf{k}^2}{4\pi c^2} \, \mathbf{q_k}, \qquad \dot{\mathbf{q}}_\mathbf{k}^* = -\frac{\delta H}{\delta \mathbf{p_k^*}} = -4\pi c^2 \, \mathbf{p_k} \tag{39}$$

Now we can see that the correct frequency results, all oscillate at $\omega_\mathbf{k}$. For example,

$$\ddot{\mathbf{p}}_\mathbf{k} = \frac{\omega_\mathbf{k}^2}{4\pi c^2} \, \dot{\mathbf{q}}_\mathbf{k}^* = -\omega_\mathbf{k}^2 \mathbf{p_k} \qquad \ddot{\mathbf{q}}_\mathbf{k} = -4\pi c^2 \, \dot{\mathbf{p}}_\mathbf{k}^* = -\omega_\mathbf{k}^2 \mathbf{q_k} \tag{40}$$

There are tricky steps in how to do the algebra correctly. Once worked through, we find that the basic modes oscillate at the frequency required by the light wave dispersion relation, $\omega_\mathbf{k} = ck$.

1.2 Quantization of modes: Simple harmonic oscillator example

Next the quantization of each mode needs to be accomplished. But since each mode analogous to a harmonic oscillator, as we'll show, the quantization is not too difficult. We already can see that the modes are independent. So proceed essentially on the individual modes, at a given wave vector and polarization. But I won't for now be writing any polarization indices, for simplicity.

Recall the quantization of a simple harmonic oscillator. The Hamiltonian can be re-expressed in some rescaled operators:

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2 = \frac{\hbar\omega}{2}\left(P^2 + Q^2\right); \quad Q = \sqrt{\frac{m\omega}{\hbar}}q, \ \ P = \frac{1}{\sqrt{m\hbar\omega}}p \tag{41}$$

Then if the original commutator is $[x, p] = i\hbar$, we have a unit commutator here,

$$[Q, P] = \sqrt{\frac{m\omega}{\hbar}}\frac{1}{\sqrt{m\hbar\omega}}[x, p] = i \tag{42}$$

The Hamiltonian can be expressed in a symmetrized form as follows:

$$H = \frac{\hbar\omega}{2}\frac{1}{2}\left[(Q + iP)(Q - iP) + (Q - iP)(Q + iP)\right] \tag{43}$$

This suggest defining the annihilation and creation operators

$$a = \frac{1}{\sqrt{2}}(Q + iP), \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) \tag{44}$$

Their commutation relation is then conveniently unity,

$$[a, a^\dagger] = (\frac{1}{\sqrt{2}})^2[Q + iP, Q - iP] = \frac{1}{2}\left\{-i[Q, P] + i[P, Q]\right\} = 1 \tag{45}$$

The coordinate and momentum are expressed

$$Q = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad P = \frac{1}{i\sqrt{2}}(a - a^\dagger). \tag{46}$$

The Hamiltonian becomes

$$H = \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a) = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{47}$$

where the last step used the commutation relation in the form, $aa^\dagger = a^\dagger a + 1$. The operator $\hat{n} = a^\dagger a$ is the number operator that counts the quanta of excitation. The number operator can be easily shown to have the following commutation relations:

$$[n, a] = [a^\dagger a, a] = [a^\dagger, a]a = -a, \quad [n, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger[a, a^\dagger] = +a, \tag{48}$$

These show that $a^\dagger$ creates or adds one quantum of excitation to the system, while a destroys or removes one quantum. The Hamiltonian famously shows how the system has a zero-point energy of $\hbar\omega/2$ and each quantum of excitation adds an additional $\hbar\omega$ of energy.

The eigenstates of the number operator $\hat{n} = a^\dagger a$ are also eigenstates of H. And while a and $a^\dagger$ lower and raise the number of quanta present, the eigenstates of the Hamiltonian are not their eigenstates. But later we need some matrix elements, hence it is good to summarize here exactly the operations of a or $a^\dagger$ on the number eigenstates, $|n\rangle$, which are assumed to be unit normalized.

If a state $|n\rangle$ is a normalized eigenstate of $\hat{n}$, with eigenvalue n, then we must have

$$a^\dagger|n\rangle = c_n|n+1\rangle, \quad \langle n|a = c_n^*\langle n+1| \tag{49}$$

where c_n is a normalization constant. Putting these together, and using the commutation relation, gives

$$1 = \langle n|aa^\dagger|n\rangle = |c_n|^2\langle n+1|n+1\rangle \Longrightarrow |c_n|^2 = \langle n|aa^\dagger|n\rangle = \langle n|a^\dagger a + 1|n\rangle = n + 1 \tag{50}$$

In the same fashion, consider the action of the lowering operator,

$$a|n\rangle = d_n|n-1\rangle, \qquad \langle n|a^\dagger = d_n^*\langle n-1| \tag{51}$$

$$1 = \langle n|a^\dagger a|n\rangle = |d_n|^2\langle n-1|n-1\rangle \implies |d_n|^2 = \langle n|a^\dagger a|n\rangle = n \tag{52}$$

Therefore when these operators act, they change the normalization slightly, and we can write

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle, \qquad a|n\rangle = \sqrt{n}\,|n-1\rangle. \tag{53}$$

Indeed, the first of these can be iterated on the ground state $|0\rangle$ that has no quanta, to produce any excited state:

$$|n\rangle = \frac{1}{\sqrt{n!}}\,(a^\dagger)^n|0\rangle \tag{54}$$

Based on these relations, then it is easy to see the basic matrix elements,

$$\langle n+1|a^\dagger|n\rangle = \sqrt{n+1}, \qquad \langle n-1|a|n\rangle = \sqrt{n}. \tag{55}$$

An easy way to remember these, is that the factor in the square root is always the larger of the number of quanta in either initial or final states. These will be applied later.

1.3 Fundamental commutation relations for the EM modes

Now how to relate what we know to the EM field Hamiltonian, Eqn. (34)? The main difference there is the presence of operators together with their complex conjugates in the classical Hamiltonian. How to decide their fundamental commutators? That is based on the fundamental commutation relation in real space (for one component only):

$$[A_i(\mathbf{r},t), p_i(\mathbf{r}',t)] = i\hbar\,\delta(\mathbf{r}-\mathbf{r}'). \tag{56}$$

The fields are expressed as in Eqns. (19) and (27). Using these expressions to evaluate the LHS of (56),

$$[A_i(\mathbf{r},t), p_i(\mathbf{r}',t)] = \frac{1}{4V}\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\left[\mathbf{A_k}e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{A_k^*}e^{-i\mathbf{k}\cdot\mathbf{r}}, \mathbf{p_{k'}}e^{i\mathbf{k}'\cdot\mathbf{r}'} + \mathbf{p_{k'}^*}e^{-i\mathbf{k}'\cdot\mathbf{r}'}\right] \tag{57}$$

In a finite volume, however, the following is a representation of a delta function:

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{V}\sum_{\mathbf{k}}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \tag{58}$$

Although not a proof, we can see that (56) and (57) match if the mode operators have the following commutation relations (for each component):

$$[\mathbf{A_k}, \mathbf{p_{k'}^*}] = i\hbar\delta_{\mathbf{k},\mathbf{k}'}, \quad [\mathbf{A_k^*}, \mathbf{p_{k'}}] = i\hbar\delta_{\mathbf{k},\mathbf{k}'}, \quad [\mathbf{A_k}, \mathbf{p_{k'}}] = i\hbar\delta_{\mathbf{k},-\mathbf{k}'}, \quad [\mathbf{A_k^*}, \mathbf{p_{k'}^*}] = i\hbar\delta_{\mathbf{k},-\mathbf{k}'}. \tag{59}$$

These together with the delta function representation, give the result for the RHS of (57),

$$[A_i(\mathbf{r},t), p_i(\mathbf{r}',t)] = \frac{1}{4V}\sum_{\pm\mathbf{k}}i\hbar\left[2e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} + 2e^{i-\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\right] = i\hbar\,\delta(\mathbf{r}-\mathbf{r}'). \tag{60}$$

Thus the basic commutators of the modes are those in (59). Now we can apply them to quantize the EM fields.

1.4 The quantization of the EM fields

At some point, one should keep in mind that these canonical coordinates are effectively scalars, once the polarization is accounted for:

$$\mathbf{q_k} = \sum_\alpha \hat{\epsilon}_k q_{\mathbf{k}\alpha}, \quad \mathbf{p_k} = \sum_\alpha \hat{\epsilon}_k p_{\mathbf{k}\alpha}. \tag{61}$$

The polarizations are decoupled, so mostly its effects can be ignored. But then the Hamiltonian (34) really has two terms at each wavevector, one for each polarization. For simplicity I will not be writing the polarization index, but just write scalar $q_\mathbf{k}$ and $p_\mathbf{k}$ for each mode's coordinate and momentum. For any scalar coordinate and its momentum, we postulate from (59)

$$[q_\mathbf{k}, p_{\mathbf{k}'}^\dagger] = i\hbar\delta_{\mathbf{k},\mathbf{k}'}, \quad [q_\mathbf{k}^\dagger, p_{\mathbf{k}'}] = i\hbar\delta_{\mathbf{k},\mathbf{k}'}, \quad [q_\mathbf{k}, p_{\mathbf{k}'}] = i\hbar\delta_{\mathbf{k},-\mathbf{k}'}, \quad [q_\mathbf{k}^\dagger, p_{\mathbf{k}'}^\dagger] = i\hbar\delta_{\mathbf{k},-\mathbf{k}'}, \tag{62}$$

Note that the first two look to be inconsistent if you think of the adjoint operation as just complex conjugate. But they are correct. Noting that $(AB^\dagger)^\dagger = BA^\dagger$, we have

$$[A, B^\dagger]^\dagger = BA^\dagger - A^\dagger B = [B, A^\dagger] = -[A^\dagger, B] \tag{63}$$

Then applied to the problem with $A = q_\mathbf{k}$ and $B = p_{\mathbf{k}'}$

$$[q_\mathbf{k}^\dagger, p_{\mathbf{k}'}] = -[q_\mathbf{k}, p_{\mathbf{k}'}^\dagger]^\dagger = -(i\hbar\delta_{\mathbf{k},\mathbf{k}'})^\dagger = i\hbar\delta_{\mathbf{k},\mathbf{k}'} \tag{64}$$

So although the relations look unsual, they are correct.

Let's look at some algebra that hopefully leads to creation and annihilation operators. First, get some coordinates and momenta with unit normalized commutators. Suppose that a given mode $\mathbf{k}\alpha$ has a Hamiltonian from (34). Consider first one term at one wave vector: [Even though, classically, the terms at $\mathbf{k}$ and $-\mathbf{k}$ in the sum give equal contributions] Consider making a transformation to $Q_\mathbf{k}$ and $P_\mathbf{k}$,

$$H_{+\mathbf{k}\alpha} = 2\pi c^2 p_\mathbf{k}^\dagger p_\mathbf{k} + \frac{\omega_\mathbf{k}^2}{8\pi c^2} q_\mathbf{k}^\dagger q_\mathbf{k} = \frac{\hbar\omega_\mathbf{k}}{2}\left(P_\mathbf{k}^\dagger P_\mathbf{k} + Q_\mathbf{k}^\dagger Q_\mathbf{k}\right) \tag{65}$$

Here because it is a quantum problem, we suppose that the terms from $\mathbf{k}$ and $-\mathbf{k}$ modes really are not the same. Thus there is a similar term for the negative wave vector:

$$H_{-\mathbf{k}\alpha} = 2\pi c^2 p_{-\mathbf{k}}^\dagger p_{-\mathbf{k}} + \frac{\omega_\mathbf{k}^2}{8\pi c^2} q_{-\mathbf{k}}^\dagger q_{-\mathbf{k}} = \frac{\hbar\omega_\mathbf{k}}{2}\left(P_{-\mathbf{k}}^\dagger P_{-\mathbf{k}} + Q_{-\mathbf{k}}^\dagger Q_{-\mathbf{k}}\right) \tag{66}$$

The right hand sides are the same as the energies for SHO's in the normalized coordinates and momenta. However, we have relations like $q_{-\mathbf{k}}^\dagger = q_\mathbf{k}$, and $q_{-\mathbf{k}} = q_\mathbf{k}^\dagger$, and we suppose they should apply to the new rescaled coordinates and momenta. So this latter relation also takes the form

$$H_{-\mathbf{k}\alpha} = 2\pi c^2 p_\mathbf{k} p_\mathbf{k}^\dagger + \frac{\omega_\mathbf{k}^2}{8\pi c^2} q_\mathbf{k} q_\mathbf{k}^\dagger = \frac{\hbar\omega_\mathbf{k}}{2}\left(P_\mathbf{k} P_\mathbf{k}^\dagger + Q_\mathbf{k} Q_\mathbf{k}^\dagger\right) \tag{67}$$

In the quantum problem, the order in which conjugate operators act is important and should not be modified. So $H_{+\mathbf{k}\alpha}$ and $H_{-\mathbf{k}\alpha}$ are not the same. To match the sides, try the identifications

$$P_\mathbf{k} = \sqrt{\frac{4\pi c^2}{\hbar\omega_\mathbf{k}}} p_\mathbf{k}, \quad Q_\mathbf{k} = \sqrt{\frac{\omega_\mathbf{k}}{4\pi c^2 \hbar}} q_\mathbf{k} \tag{68}$$

The basic commutator that results between them is now unit normalized,

$$[Q_\mathbf{k}, P_\mathbf{k}^\dagger] = \sqrt{\frac{\omega_\mathbf{k}}{4\pi c^2 \hbar}}\sqrt{\frac{4\pi c^2}{\hbar\omega_\mathbf{k}}}[q_\mathbf{k}, p_\mathbf{k}^\dagger] = \frac{1}{\hbar}i\hbar = i \tag{69}$$

It is obvious one can also show

$$[Q_\mathbf{k}^\dagger, P_\mathbf{k}] = i \tag{70}$$

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Now we can re-express the energy in the sense of operators like what was done for the SHO, although it is more complicated here because of the two directions for the wavevectors. Note the following algebra that results if we try that, for complex operators:

$$
\begin{aligned}
F_1 &= \frac{1}{2}\left[(Q_{\mathbf{k}} + iP_{\mathbf{k}})(Q_{\mathbf{k}}^\dagger - iP_{\mathbf{k}}^\dagger) + (Q_{\mathbf{k}}^\dagger - iP_{\mathbf{k}}^\dagger)(Q_{\mathbf{k}} + iP_{\mathbf{k}})\right] \\
&= Q_{\mathbf{k}}^\dagger Q_{\mathbf{k}} + P_{\mathbf{k}}^\dagger P_{\mathbf{k}} + \frac{i}{2}\left[P_{\mathbf{k}}Q_{\mathbf{k}}^\dagger + Q_{\mathbf{k}}^\dagger P_{\mathbf{k}} - P_{\mathbf{k}}^\dagger Q_{\mathbf{k}} - Q_{\mathbf{k}}P_{\mathbf{k}}^\dagger\right]
\end{aligned}
\tag{71}
$$

That has extra terms that we do not want. To get rid of them, consider also the contribution from the opposite wave vector. We use the same form, but with $-\mathbf{k}$, and employing $Q_{-\mathbf{k}} = Q_{\mathbf{k}}^\dagger$, $P_{-\mathbf{k}} = P_{\mathbf{k}}^\dagger$.

$$
\begin{aligned}
F_2 &= \frac{1}{2}\left[(Q_{-\mathbf{k}} + iP_{-\mathbf{k}})(Q_{-\mathbf{k}}^\dagger - iP_{-\mathbf{k}}^\dagger) + (Q_{-\mathbf{k}}^\dagger - iP_{-\mathbf{k}}^\dagger)(Q_{-\mathbf{k}} + iP_{-\mathbf{k}})\right] \\
&= \frac{1}{2}\left[(Q_{\mathbf{k}}^\dagger + iP_{\mathbf{k}}^\dagger)(Q_{\mathbf{k}} - iP_{\mathbf{k}}) + (Q_{\mathbf{k}} - iP_{\mathbf{k}})(Q_{\mathbf{k}}^\dagger + iP_{\mathbf{k}}^\dagger)\right] \\
&= Q_{\mathbf{k}}^* Q_{\mathbf{k}} + P_{\mathbf{k}}^* P_{\mathbf{k}} - \frac{i}{2}\left[P_{\mathbf{k}}Q_{\mathbf{k}}^\dagger + Q_{\mathbf{k}}^\dagger P_{\mathbf{k}} - P_{\mathbf{k}}^\dagger Q_{\mathbf{k}} - Q_{\mathbf{k}}P_{\mathbf{k}}^\dagger\right]
\end{aligned}
\tag{72}
$$

So the combination of the two expressions eliminates the imaginary part, leaving only the part we want in the Hamiltonian. Therefore, algebraically speaking we can write:

$$
H_{+\mathbf{k}\alpha} = \frac{\hbar\omega_{\mathbf{k}}}{2}\frac{1}{2}(F_1 + F_2)
\tag{73}
$$

Based on these expressions, introduce creation and annihilation operators, for both the positive and negative wave vectors:

$$
a_{\mathbf{k}} = \frac{1}{\sqrt{2}}(Q_{\mathbf{k}} + iP_{\mathbf{k}}), \qquad a_{\mathbf{k}}^\dagger = \frac{1}{\sqrt{2}}(Q_{\mathbf{k}}^\dagger - iP_{\mathbf{k}}^\dagger).
\tag{74}
$$

$$
a_{-\mathbf{k}} = \frac{1}{\sqrt{2}}(Q_{\mathbf{k}}^\dagger + iP_{\mathbf{k}}^\dagger), \qquad a_{-\mathbf{k}}^\dagger = \frac{1}{\sqrt{2}}(Q_{\mathbf{k}} - iP_{\mathbf{k}}).
\tag{75}
$$

By their definitions, they must have unit real commutators, e.g.,

$$
[a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger] = \frac{1}{\sqrt{2}^2}[Q_{\mathbf{k}} + iP_{\mathbf{k}}, Q_{\mathbf{k}}^\dagger - iP_{\mathbf{k}}^\dagger] = \frac{1}{2}\left\{-i[Q_{\mathbf{k}}, P_{\mathbf{k}}^\dagger] + i[P_{\mathbf{k}}, Q_{\mathbf{k}}^\dagger]\right\} = 1
\tag{76}
$$

On the other hand, a commutator between different modes (or with different polarizations at one wave vector) gives zero. The individual term in the Hamiltonian sum is

$$
H_{+\mathbf{k}\alpha} = \frac{\hbar\omega_{\mathbf{k}}}{2}\frac{1}{2}\left[a_{\mathbf{k}}a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}}a_{-\mathbf{k}}^\dagger + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}}\right]
\tag{77}
$$

So the total field Hamiltonian is the sum

$$
H = \sum_{\mathbf{k}} \frac{\hbar\omega_{\mathbf{k}}}{2}\left[a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + 1/2 + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}} + 1/2\right]
\tag{78}
$$

The sum is over all wave vectors, and the positive and negative terms give the same total, so

$$
H = \sum_{\mathbf{k}\alpha} \hbar\omega_{\mathbf{k}}\left(a_{\mathbf{k}\alpha}^\dagger a_{\mathbf{k}\alpha} + \frac{1}{2}\right) = \sum_{\mathbf{k}\alpha}\left(n_{\mathbf{k}\alpha} + \frac{1}{2}\right)
\tag{79}
$$

The number operator is implicitly defined here:

$$
n_{\mathbf{k}\alpha} = a_{\mathbf{k}\alpha}^\dagger a_{\mathbf{k}\alpha}
\tag{80}
$$

Then each mode specified by a wave vector and a polarization contributes $\hbar\omega(a_{\mathbf{k}\alpha}^\dagger a_{\mathbf{k}\alpha} + 1/2)$ to the Hamiltonian. Every mode is equivalent, mathematically, to a simple harmonic oscillator. What

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could be more simple? Really, it is hard to believe, when you think about it. The modes are completely independent, at this level, they do not interfere. There is just a linear superposition of their EM fields. To get those fields, summarize a few more relationships.

The fields associated with the creation and annihilation operators are found via solving their definitions,

$$Q_{\mathbf{k}} = \frac{1}{\sqrt{2}}(a_{\mathbf{k}} + a_{-\mathbf{k}}^{\dagger}), \quad P_{\mathbf{k}} = \frac{1}{i\sqrt{2}}(a_{\mathbf{k}} - a_{-\mathbf{k}}^{\dagger}), \tag{81}$$

Then with

$$\mathbf{A}_{\mathbf{k}\alpha} = A_{\mathbf{k}\alpha}\hat{\epsilon}_{\mathbf{k}\alpha}, \quad A_{\mathbf{k}\alpha} = q_{\mathbf{k}\alpha} = \sqrt{\frac{4\pi c^2 \hbar}{\omega_{\mathbf{k}}}} Q_{\mathbf{k}\alpha}, \tag{82}$$

applied into (19), the quantized fields are determined,

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{2\sqrt{V}} \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}\alpha} \left\{ \sqrt{\frac{4\pi c^2 \hbar}{\omega_{\mathbf{k}}}} \frac{(a_{\mathbf{k}\alpha} + a_{-\mathbf{k}\alpha}^{\dagger})}{\sqrt{2}} e^{i\mathbf{k}\cdot\mathbf{r}} + \sqrt{\frac{4\pi c^2 \hbar}{\omega_{\mathbf{k}}}} \frac{(a_{-\mathbf{k}\alpha} + a_{\mathbf{k}\alpha}^{\dagger})}{\sqrt{2}} e^{-i\mathbf{k}\cdot\mathbf{r}} \right\} \tag{83}$$

Swapping some terms between the positive and negative wave vector sums, this is the same as

$$\mathbf{A}(\mathbf{r},t) = \sqrt{\frac{2\pi c^2 \hbar}{V}} \sum_{\mathbf{k}} \frac{\hat{\epsilon}_{\mathbf{k}\alpha}}{\sqrt{\omega_{\mathbf{k}}}} \left[a_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\mathbf{r}} + a_{\mathbf{k}\alpha}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{84}$$

Then the vector potential determines both the electric and magnetic fields by (20) and (21), which give

$$\mathbf{E}(\mathbf{r},t) = i\sqrt{\frac{2\pi \hbar}{V}} \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}\hat{\epsilon}_{\mathbf{k}\alpha}}{\sqrt{\omega_{\mathbf{k}}}} \left[a_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k}\alpha}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{85}$$

$$\mathbf{B}(\mathbf{r},t) = i\sqrt{\frac{2\pi \hbar}{V}} \sum_{\mathbf{k}} \frac{c\mathbf{k} \times \hat{\epsilon}_{\mathbf{k}\alpha}}{\sqrt{\omega_{\mathbf{k}}}} \left[a_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k}\alpha}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{86}$$

Indeed, after all this work, the fields have a certain simplicity. Their amplitude depends on Planck's constant. Thus there must be quantum fluctuations determined by it.

The above do not show the explicit time dependence. However, that is implicit in the creation/annihilation operators. Based on the Hamiltonian, their equations of motion are simple:

$$i\hbar \dot{a}_{\mathbf{k}\alpha} = [a_{\mathbf{k}\alpha}, H] = \hbar\omega_{\mathbf{k}}[a_{\mathbf{k}}, a_{\mathbf{k}\alpha}^{\dagger} a_{\mathbf{k}\alpha}] = \hbar\omega_{\mathbf{k}}[a_{\mathbf{k}\alpha}, a_{\mathbf{k}\alpha}^{\dagger}]a_{\mathbf{k}\alpha} = +\hbar\omega_{\mathbf{k}} a_{\mathbf{k}\alpha} \tag{87}$$

$$i\hbar \dot{a}_{\mathbf{k}\alpha}^{\dagger} = [a_{\mathbf{k}\alpha}^{\dagger}, H] = \hbar\omega_{\mathbf{k}}[a_{\mathbf{k}\alpha}^{\dagger}, a_{\mathbf{k}\alpha}^{\dagger} a_{\mathbf{k}\alpha}] = \hbar\omega_{\mathbf{k}} a_{\mathbf{k}\alpha}^{\dagger}[a_{\mathbf{k}\alpha}^{\dagger} a_{\mathbf{k}\alpha}] = -\hbar\omega_{\mathbf{k}} a_{\mathbf{k}\alpha}^{\dagger} \tag{88}$$

And then they oscillate at opposite frequencies:

$$\dot{a}_{\mathbf{k}\alpha} = -i\omega_{\mathbf{k}} a_{\mathbf{k}\alpha} \implies a_{\mathbf{k}\alpha}(t) = a_{\mathbf{k}\alpha}(0)e^{-i\omega_{\mathbf{k}}t} \tag{89}$$

$$\dot{a}_{\mathbf{k}\alpha}^{\dagger} = +i\omega_{\mathbf{k}} a_{\mathbf{k}\alpha}^{\dagger} \implies a_{\mathbf{k}\alpha}^{\dagger}(t) = a_{\mathbf{k}\alpha}^{\dagger}(0)e^{+i\omega_{\mathbf{k}}t} \tag{90}$$

1.5 Quantized field properties: momentum, angular momentum

The fields not only carry energy, but it is directed, so they carry linear momentum and even angular momentum. The linear momentum is

$$\mathbf{G} = \int d^3\mathbf{r} \, \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} = \frac{i^2}{4\pi c} \frac{hc}{V} \sum_{\mathbf{k}\alpha} \sum_{\mathbf{k}'\alpha'} [\hat{\epsilon}_{\mathbf{k}\alpha} \times (\mathbf{k}' \times \hat{\epsilon}_{\mathbf{k}'\alpha'})] \times$$

$$\int d^3\mathbf{r} \, (a_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k}\alpha}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{r}})(a_{\mathbf{k}'\alpha'} e^{i\mathbf{k}'\cdot\mathbf{r}} - a_{\mathbf{k}'\alpha'}^{\dagger} e^{-i\mathbf{k}'\cdot\mathbf{r}}) \tag{91}$$

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The orthognality relation only gives nonzero terms where $\mathbf{k}' = \mathbf{k}$ and $\mathbf{k}' = -\mathbf{k}$, and there results

$$\mathbf{G} = \frac{hc}{4\pi} \sum_{\mathbf{k}\alpha} \sum_{\mathbf{k}'\alpha'} [\hat{\epsilon}_{\mathbf{k}\alpha} \times (\mathbf{k}' \times \hat{\epsilon}_{\mathbf{k}'\alpha'})] \left\{ (a_{\mathbf{k}\alpha} a_{\mathbf{k}'\alpha'}^\dagger + a_{\mathbf{k}\alpha}^\dagger a_{\mathbf{k}'\alpha'})\delta_{\mathbf{k},\mathbf{k}'} - (a_{\mathbf{k}\alpha} a_{\mathbf{k}'\alpha'} + a_{\mathbf{k}\alpha}^\dagger a_{\mathbf{k}'\alpha'}^\dagger)\delta_{\mathbf{k},-\mathbf{k}'} \right\} \quad (92)$$

The orthogonality of the polarization vectors with each other and with $\mathbf{k}$ forces that vector cross product to be just $\mathbf{k}\delta_{\alpha,\alpha'}$. Only the direct terms give a nonzero result:

$$\mathbf{G} = \frac{hc}{4\pi} \sum_{\mathbf{k}\alpha} \mathbf{k}(a_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha}^\dagger + a_{\mathbf{k}\alpha}^\dagger a_{\mathbf{k}\alpha}) = \sum_{\mathbf{k}\alpha} \hbar c \mathbf{k} \, a_{\mathbf{k}\alpha}^\dagger a_{\mathbf{k}\alpha} \quad (93)$$

There is no zero point term, because of the cancellation between a term at $+\mathbf{k}$ and one at $-\mathbf{k}$. Then each mode carries a linear momentum of $\hbar c \mathbf{k}$.

Consider the angular momentum. The contribution from the fields is

$$\mathbf{J} = \frac{1}{4\pi c} \int d^3\mathbf{r} \, [\mathbf{r} \times (\mathbf{E} \times \mathbf{B})] \quad (94)$$

But there is the identity,

$$[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})] = \mathbf{E}(\mathbf{r} \cdot \mathbf{B}) - (\mathbf{r} \cdot \mathbf{E})\mathbf{B}. \quad (95)$$

Furthermore, for any mode, the wave vector is perpendicular to both $\mathbf{E}$ and $\mathbf{B}$. So this definition of angular momentum seems to give zero for the total component along the direction of propagation. Even if it more properly symmetrized for the QM problem, it still gives zero.

That shows that the concept of angular momentum in an EM field is tricky. Possibly, assuming a plane wave is too restrictive, and instead one should not make any particular assumption on the nature of the fields, to start with. One can do a more careful analysis, that shows the angular momentum is composed from an orbital part and a spin part.

Consider the following vector algebra for the ith component of the argument in the angular momentum integral (essentially, the angular momentum density). Here the Levi-Civita symbol is used for the cross products, and the magnetic field is expressed via the vector potential. Repeated indeces are summer over.

$$\begin{aligned}
[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})]_i &= \varepsilon_{ijk} x_j (\mathbf{E} \times \mathbf{B})_k = \varepsilon_{ijk} x_j \, \varepsilon_{klm} E_l B_m \\
&= \varepsilon_{ijk} x_j \, \varepsilon_{klm} E_l \, (\varepsilon_{mnp} \partial_n A_p) \\
&= \varepsilon_{ijk} x_j \, E_l (\delta_{kn}\delta_{lp} - \delta_{kp}\delta_{ln}) \partial_n A_p \\
&= \varepsilon_{ijk} x_j \, [E_l \partial_k A_l - E_l \partial_l A_k] \quad (96)
\end{aligned}$$

Now when this is integrated over all space, the last term can be integrated by parts, dropping any vanishing surface terms at infinity. Further, far from any sources, the electric field is divergenceless, so $\vec{\nabla} \cdot \mathbf{E} = \partial_l E_l = 0$. So now the expression becomes

$$\begin{aligned}
[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})]_i &= \varepsilon_{ijk} [x_j E_l \partial_k A_l + \partial_l(x_j E_l) A_k] = \varepsilon_{ijk} [x_j E_l \partial_k A_l + \delta_{lj} E_l A_k] \\
&= \varepsilon_{ijk} [x_j E_l \partial_k A_l + E_j A_k] = E_l(\varepsilon_{ijk} x_j \partial_k) A_l + \varepsilon_{ijk} E_j A_k \quad (97)
\end{aligned}$$

This is an interesting expression. The first term contains effectively the orbital angular momentum operator acting between $\mathbf{E}$ and $\mathbf{A}$. The second term is their cross product. Then the total angular momentum integrated over space is

$$\mathbf{J} = \frac{1}{4\pi c} \int d^3\mathbf{r} \, \{E_l(\mathbf{r} \times \nabla)A_l + \mathbf{E} \times \mathbf{A}\} \quad (98)$$

Both terms can be written as operators acting between the fields, adding $\hbar$ in appropriate places:

$$[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})]_i = \frac{i}{\hbar} \{E_l(-i\hbar \, \varepsilon_{ijk} x_j \partial_k)A_l + E_j(-i\hbar \, \varepsilon_{ijk})A_k\} \quad (99)$$

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The first term in (99) contains the orbital angular momentum operator, $\mathbf{L}_i = (\mathbf{r} \times \mathbf{p})_i$, acting on identical components of $\mathbf{E}$ and $\mathbf{A}$; it is a diagonal operator. The second contains what is a spin-1 operator, for which one can write its ith component,

$$(S_i)_{jk} = -i\hbar\,\varepsilon_{ijk}, \qquad (\mathbf{S})_{jk} = (S_i \hat{x}_i)_{jk} = -i\hbar\,\hat{x}_i \varepsilon_{ijk}. \tag{100}$$

Then the total angular momentum is expressed as a sum of these two parts, each being matrix elements of an operator,

$$\mathbf{J} = \frac{i}{4\pi\hbar c} \int d^3\mathbf{r}\; E_j \left[(\mathbf{r} \times \mathbf{p})\delta_{jk} + (\mathbf{S})_{jk} \right] A_k \tag{101}$$

Well, really $(S_i)_{jk}$ comes from the cross product operator, however, it can be seen to be a quantum spin operator, that couples different components of $\mathbf{E}$ and $\mathbf{A}$. This operator is defined here by its matrices, one for each component i, where j and k are the column and row

$$\begin{array}{c} (S_i)_{jk} \\ (i=x,y,z) \end{array} = -i\hbar\varepsilon_{ijk} = -i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, -i\hbar \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, -i\hbar \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{102}$$

Check some properties to be convinced that this is really a spin-1 operator. Consider a commutator between two of these, using the properties of the Levi-Civita symbol. Start from the matrix multiplications, giving the ln^{th} element of the matrix products:

$$[(S_i)(S_j)]_{ln} = (S_i)_{lm}(S_j)_{mn} = (-i\hbar)^2 \varepsilon_{ilm}\,\varepsilon_{jmn} = (-i\hbar)^2 (\delta_{in}\delta_{lj} - \delta_{ij}\delta_{ln}) \tag{103}$$

$$[(S_j)(S_i)]_{ln} = (S_j)_{lm}(S_i)_{mn} = (-i\hbar)^2 \varepsilon_{jlm}\,\varepsilon_{imn} = (-i\hbar)^2 (\delta_{jn}\delta_{li} - \delta_{ji}\delta_{ln}) \tag{104}$$

The difference cancels the last terms,

$$[(S_i)(S_j) - (S_j)(S_i)]_{ln} = (-i\hbar)^2 (\delta_{in}\delta_{lj} - \delta_{jn}\delta_{li}) \tag{105}$$

Then the difference of deltas can be put back into a product of ε's.

$$[(S_i)(S_j) - (S_j)(S_i)]_{ln} = (-i\hbar)^2 (-\varepsilon_{ijk})\varepsilon_{lnk} = i\hbar\varepsilon_{ijk}[-i\hbar\varepsilon_{kln}] = i\hbar\varepsilon_{ijk}(S_k)_{ln}. \tag{106}$$

Therefore these matrices do have the commutation relations for an angular momentum,

$$[(S_i),(S_j)] = i\hbar\,\varepsilon_{ijk}\,(S_k). \tag{107}$$

Also, look at the matrix of $\vec{S}^2$, within the space that the operators act:

$$(\vec{S}^2)_{ln} = (S_i)_{lm}(S_i)_{mn} = (-i\hbar)^2 \varepsilon_{ilm}\varepsilon_{imn} = (-i\hbar)^2 (\delta_{lm}\delta_{mn} - \delta_{ln}\delta_{mm}) \tag{108}$$

The expression is summed over $m = x, y, z$, both the terms are diagonal. But $\delta_{lm}\delta_{mn} = \delta_{ln}$, while $\delta_{mm} = 3$. Then this square is the diagonal matrix:

$$(\vec{S}^2)_{ln} = (-i\hbar)^2 (\delta_{ln} - 3\delta_{ln}) = 2\hbar^2 \delta_{ln}. \tag{109}$$

This clearly has $s(s+1) = 2$ with $s = 1$, so indeed it corresponds to spin-1.

It may seem curious, that none of the matrices are diagonal. But this just means that the Cartesian axes, to which these correspond, are not the good quantization axes. For example, find the eigenvectors of the operator (S_x). The eigenvalue problem is

$$(S_x - \lambda I)(u) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -i\hbar \\ 0 & i\hbar & -\lambda \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = 0. \tag{110}$$

The eigenvalues are obviously $\lambda = 0, \pm\hbar$. The $s_x = 0$ eigenvector is trivial, $u = (1, 0, 0)$, and seems to have little physical importance. The $s_x = +\hbar$ eigenvector is $u = \frac{1}{\sqrt{2}}(0, 1, i)$ and the

$s_x = -\hbar$ eigenvector is $u = \frac{1}{\sqrt{2}}(0, 1, -i)$. If the vector potential were expressed in these as a basis, the spin angular momentum components along x are specified. But this basis is not the pure Cartesian components. It requires linear combinations of Cartesian components out of phase by $\pm 90°$. These combinations are states of circular polarization, which are the "good" states of spin angular momentum. Thus, none of these three matrices is diagonal when expressed in Cartesian components. More on the spin angular momentum and EM-wave polarization is discussed in the next section.

Note, if one had sought the eigenvectors of the matrix (S_z), one finds they are $u = (0, 0, 1)$ for $s_z = 0$, and $u = (1, i, 0)$ for $s_z = +\hbar$, and $u = (1, -i, 0)$ for $s_z = -\hbar$. These last two correspond to states where the $\mathbf{A}$ and $\mathbf{E}$ fields are rotating around the z-axis. It is typical to consider waves propagating along z, hence, we see these vectors appear again when polarization is discussed for this wave vector direction.

Thus the second part of the angular momentum involves just the cross product of $\mathbf{E}$ and $\mathbf{A}$, which is considered the intrinsic *spin angular momentum* in the EM fields. It can seen to be the same as the canonical angular momentum in the fields, although it is hard to say in general why this is true. If one uses a definition like (coordinate $\times$ conjugate momentum), integrated over space, where the coordinate is the vector potential, and its conjugate momentum is $-\mathbf{E}/4\pi c$, one gets

$$\mathbf{S} = \int d^3\mathbf{r} \; \mathbf{A} \times \frac{-\mathbf{E}}{4\pi c} = \frac{-1}{4\pi c} \int d^3\mathbf{r} \; (\mathbf{A} \times \mathbf{E}) \tag{111}$$

Except for the ordering, it is the same as $\mathbf{S}$ obtained above. Assume the ordering doesn't matter (both are fields depending on position), and continue to evaluate it,

$$\mathbf{S} = \frac{-1}{4\pi c} \frac{ihc}{V} \sum_{\mathbf{k}\alpha} \sum_{\mathbf{k}'\alpha'} \hat{\epsilon}_{\mathbf{k}\alpha} \times \hat{\epsilon}_{\mathbf{k}'\alpha'} \int d^3\mathbf{r} \; \left[a_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\mathbf{r}} + a^\dagger_{\mathbf{k}\alpha} e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \left[a_{\mathbf{k}'\alpha'} e^{i\mathbf{k}'\cdot\mathbf{r}} - a^\dagger_{\mathbf{k}'\alpha'} e^{-i\mathbf{k}'\cdot\mathbf{r}} \right] \tag{112}$$

The integrations are the usual orthogonality relations, which give terms where $\mathbf{k}' = \mathbf{k}$ and terms where $\mathbf{k}' = -\mathbf{k}$. Only the first set gives nonzero results, due to the cross products of the polarization vectors (Take them oppositely directed for the $-\mathbf{k}$ mode compared to the $\mathbf{k}$ mode. Further, we look for a quantity whose expectation value is nonzero.) So there remains only the terms

$$\mathbf{S} = \frac{ih}{4\pi} \sum_{\mathbf{k}\alpha\alpha'} \hat{\epsilon}_{\mathbf{k}\alpha} \times \hat{\epsilon}_{\mathbf{k}\alpha'} (a_{\mathbf{k}\alpha} a^\dagger_{\mathbf{k}\alpha'} - a^\dagger_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha'}) \tag{113}$$

Here the two polarizations must be different to give a nonzero result. We suppose they are oriented in such a way that

$$\hat{\epsilon}_{\mathbf{k}1} \times \hat{\epsilon}_{\mathbf{k}2} = \hat{k} \tag{114}$$

so that this cross product is along the propagation direction. Then there are two equal terms and the net is

$$\mathbf{S} = \sum_{\mathbf{k}} i\hbar\hat{k} \; (a_{\mathbf{k}1} a^\dagger_{\mathbf{k}2} - a^\dagger_{\mathbf{k}1} a_{\mathbf{k}2}) \tag{115}$$

As shown with the eigenvalues of the S matrices, the basic unit of spin angular momentum is $\hbar$, and it has a component only along (or opposite to) the propagation direction.

1.6 Orbital angular momentum

Mostly in atomic processes, the spin angular momentum is absorbed or emitted when photons are absorbed or emitted by atoms. Not much is usually mentioned about the orbital angular momentum in the EM fields. Consider here what $\mathbf{L}$ is for the quantized EM field, using the expression,

$$\mathbf{L} = \frac{1}{4\pi c} \int d^3\mathbf{r} \; E_l (\mathbf{r} \times \nabla) A_l. \tag{116}$$

If the ∇ operation is applied to the fields in (84), it pulls out $i\mathbf{k}$ for each mode. One then has

$$(\mathbf{r} \times \nabla)A_l = \sqrt{\frac{2\pi c^2 \hbar}{V}} \sum_{\mathbf{k}} \frac{\hat{x}_l \cdot \hat{\epsilon}_{\mathbf{k}\alpha}}{\sqrt{\omega_{\mathbf{k}}}} \mathbf{r} \times (i\mathbf{k}) \left[a_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k}\alpha}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{117}$$

Now combine with the same component of the electric field,

$$\begin{aligned}
E_l(\mathbf{r} \times \nabla)A_l &= i\sqrt{\frac{2\pi\hbar}{V}} \sum_{\mathbf{k}'\alpha'} \frac{\omega_{\mathbf{k}'}\hat{x}_l \cdot \hat{\epsilon}_{\mathbf{k}'\alpha'}}{\sqrt{\omega_{\mathbf{k}'}}} \left[a_{\mathbf{k}'\alpha'} e^{i\mathbf{k}'\cdot\mathbf{r}} - a_{\mathbf{k}'\alpha'}^\dagger e^{-i\mathbf{k}'\cdot\mathbf{r}} \right] \\
&\times \sqrt{\frac{2\pi c^2 \hbar}{V}} \sum_{\mathbf{k}\alpha} \frac{\hat{x}_l \cdot \hat{\epsilon}_{\mathbf{k}\alpha}}{\sqrt{\omega_{\mathbf{k}}}} \mathbf{r} \times (i\mathbf{k}) \left[a_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k}\alpha}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}} \right]
\end{aligned} \tag{118}$$

The orbital angular momentum then is

$$\mathbf{L} = \frac{ih}{4\pi V} \sum_{\mathbf{k},\mathbf{k}',\alpha\alpha'} \sqrt{\frac{\omega_{\mathbf{k}'}}{\omega_{\mathbf{k}}}} \hat{\epsilon}_{\mathbf{k}'\alpha'} \cdot \hat{\epsilon}_{\mathbf{k}\alpha} \int d^3\mathbf{r}\, \mathbf{r} \left[a_{\mathbf{k}'\alpha'} e^{i\mathbf{k}'\cdot\mathbf{r}} - a_{\mathbf{k}'\alpha'}^\dagger e^{-i\mathbf{k}'\cdot\mathbf{r}} \right] \left[a_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k}\alpha}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \times (i\mathbf{k}) \tag{119}$$

The basic integral to evaluate here is not exactly a normalization integral:

$$I_x = \int d^3\mathbf{r}\, x e^{i\mathbf{k}'\cdot\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}} = -i\frac{\partial}{\partial k_x} \int d^3\mathbf{r}\, e^{i\mathbf{k}'\cdot\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}} = -i\frac{\partial}{\partial k_x} V\delta_{\mathbf{k},\mathbf{k}'} \tag{120}$$

It would seem to be zero, although singular in some sense. For now I'll consider that the orbital angular momentum should be zero.

One thing that can be said with more certainty is the component of $\mathbf{L}$ along the propagation direction, $\mathbf{k}$, for some mode in the sum. As $[\mathbf{r} \times (i\mathbf{k})] \cdot \mathbf{k} = 0$, for a particular mode, there is no orbital angular momentum component in the direction of propagation.

1.7 Polarization

It is better to express the spin angular momentum $\mathbf{S}$ in terms of circular polarization components. In the expressions for $\mathbf{E}$, the cartesian polarization vectors could be re-expressed in terms of rotating basis vectors. For example, consider a wave moving in the z direction, with $\hat{\epsilon}_1 = \hat{x}$ and $\hat{\epsilon}_2 = \hat{y}$. Then if you look at, for example,

$$(\hat{x} + i\hat{y})e^{-i\omega t} = (\hat{x}\cos\omega t + \hat{y}\sin\omega t) + i(-\hat{x}\sin\omega t + \hat{y}\cos\omega t) \tag{121}$$

At $t = 0$, the real part is along $\hat{x}$ and the imaginary part is along $\hat{y}$. At time progresses, both the real and imaginary parts rotate counterclockwise when viewed in the usual xy-plane. I am supposing this multiplying the positive wave, $a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}$. Then both of these rotate in the positive helicity sense, where the angular momentum is in the same direction as the wave propagation. The following wave rotates in the opposite sense, clockwise or negative helicity:

$$(\hat{x} - i\hat{y})e^{-i\omega t} = (\hat{x}\cos\omega t - \hat{y}\sin\omega t) + i(-\hat{x}\sin\omega t - \hat{y}\cos\omega t) \tag{122}$$

These suggest inventing polarization basis vectors for these two helicities (the wave vector index is suppressed),

$$\hat{\epsilon}_L = \hat{\epsilon}_+ = \frac{1}{\sqrt{2}}(\hat{\epsilon}_1 + i\hat{\epsilon}_2), \quad \hat{\epsilon}_R = \hat{\epsilon}_- = \frac{1}{\sqrt{2}}(\hat{\epsilon}_1 - i\hat{\epsilon}_2) \tag{123}$$

I use L and R for left and right in place of positive and negative. The inverse relations are

$$\hat{\epsilon}_1 = \frac{1}{\sqrt{2}}(\hat{\epsilon}_L + \hat{\epsilon}_R), \quad \hat{\epsilon}_2 = \frac{1}{i\sqrt{2}}(\hat{\epsilon}_L - \hat{\epsilon}_R) \tag{124}$$

Then we see that in the expression for the electric field, there appears a combination $\sum_\alpha \hat{\epsilon}_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha}$, or

$$\hat{\epsilon}_1 a_{\mathbf{k}1} + \hat{\epsilon}_2 a_{\mathbf{k}2} = \frac{1}{\sqrt{2}}(\hat{\epsilon}_L + \hat{\epsilon}_R)a_{\mathbf{k}1} + \frac{1}{i\sqrt{2}}(\hat{\epsilon}_L - \hat{\epsilon}_R)a_{\mathbf{k}2} = \frac{1}{\sqrt{2}}(a_{\mathbf{k}1} - ia_{\mathbf{k}2})\hat{\epsilon}_L + \frac{1}{\sqrt{2}}(a_{\mathbf{k}1} + ia_{\mathbf{k}2})\hat{\epsilon}_R \tag{125}$$

This shows the two new alternative (circularly polarized) annihilation operators,

$$a_{\mathbf{k}L} \equiv \frac{1}{\sqrt{2}}(a_{\mathbf{k}1} - ia_{\mathbf{k}2}), \qquad a_{\mathbf{k}R} \equiv \frac{1}{\sqrt{2}}(a_{\mathbf{k}1} + ia_{\mathbf{k}2}). \tag{126}$$

Their inverse relations are

$$a_{\mathbf{k}1} = \frac{1}{\sqrt{2}}(a_{\mathbf{k}L} + a_{\mathbf{k}R}), \qquad a_{\mathbf{k}2} = \frac{i}{\sqrt{2}}(a_{\mathbf{k}L} - a_{\mathbf{k}R}). \tag{127}$$

Then a sum can be over either linear or circular basis:

$$\sum_{\alpha} \hat{\epsilon}_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha} = \hat{\epsilon}_1 a_{\mathbf{k}1} + \hat{\epsilon}_2 a_{\mathbf{k}2} = \hat{\epsilon}_L a_{\mathbf{k}L} + \hat{\epsilon}_R a_{\mathbf{k}R} \tag{128}$$

Additionally there are the corresponding creation operators,

$$a_{\mathbf{k}L}^{\dagger} \equiv \frac{1}{\sqrt{2}}(a_{\mathbf{k}1}^{\dagger} + ia_{\mathbf{k}2}^{\dagger}), \qquad a_{\mathbf{k}R}^{\dagger} \equiv \frac{1}{\sqrt{2}}(a_{\mathbf{k}1}^{\dagger} - ia_{\mathbf{k}2}^{\dagger}). \tag{129}$$

Their inverse relations are

$$a_{\mathbf{k}1}^{\dagger} = \frac{1}{\sqrt{2}}(a_{\mathbf{k}L}^{\dagger} + a_{\mathbf{k}R}^{\dagger}), \qquad a_{\mathbf{k}2}^{\dagger} = \frac{-i}{\sqrt{2}}(a_{\mathbf{k}L}^{\dagger} - a_{\mathbf{k}R}^{\dagger}). \tag{130}$$

The expressions for the fields really don't depend on which basis is used. However, the ones stated earlier do need to be modified to be more general, since now the basis vectors can be complex. To be totally consistent for the creation terms, we need to satisfy the conjugate relation

$$\sum_{\alpha} \hat{\epsilon}_{\mathbf{k}\alpha}^{\dagger} a_{\mathbf{k}\alpha}^{\dagger} = \hat{\epsilon}_1^{\dagger} a_{\mathbf{k}1}^{\dagger} + \hat{\epsilon}_2^{\dagger} a_{\mathbf{k}2}^{\dagger} = \hat{\epsilon}_L^{\dagger} a_{\mathbf{k}L}^{\dagger} + \hat{\epsilon}_R^{\dagger} a_{\mathbf{k}R}^{\dagger} \tag{131}$$

It means that the correct expressions for the fields in the case of complex basis vectors must be

$$\mathbf{A}(\mathbf{r}, t) = \sqrt{\frac{2\pi c^2 \hbar}{V}} \sum_{\mathbf{k}} \frac{1}{\sqrt{\omega_{\mathbf{k}}}} \left[\hat{\epsilon}_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{\epsilon}_{\mathbf{k}\alpha}^{\dagger} a_{\mathbf{k}\alpha}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{132}$$

$$\mathbf{E}(\mathbf{r}, t) = i\sqrt{\frac{2\pi\hbar}{V}} \sum_{\mathbf{k}} \sqrt{\omega_{\mathbf{k}}} \left[\hat{\epsilon}_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{\epsilon}_{\mathbf{k}\alpha}^{\dagger} a_{\mathbf{k}\alpha}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{133}$$

$$\mathbf{B}(\mathbf{r}, t) = i\sqrt{\frac{2\pi\hbar}{V}} \sum_{\mathbf{k}} \frac{c\mathbf{k}\times}{\sqrt{\omega_{\mathbf{k}}}} \left[\hat{\epsilon}_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{\epsilon}_{\mathbf{k}\alpha}^{\dagger} a_{\mathbf{k}\alpha}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \tag{134}$$

The combinations of operators with their conjugates shows that these totals are hermitian.

Now the expression for the spin angular momentum can be expressed using the circular components,

$$\mathbf{S} = \sum_{\mathbf{k}} i\hbar\hat{k} \left[\frac{1}{\sqrt{2}}(a_{\mathbf{k}L} + a_{\mathbf{k}R}) \frac{-i}{\sqrt{2}}(a_{\mathbf{k}L}^{\dagger} - a_{\mathbf{k}R}^{\dagger}) - \frac{1}{\sqrt{2}}(a_{\mathbf{k}L}^{\dagger} + a_{\mathbf{k}R}^{\dagger}) \frac{i}{\sqrt{2}}(a_{\mathbf{k}L} - a_{\mathbf{k}R}) \right] \tag{135}$$

The different polarizations commute, and the only nonzero commutation relations for the circular polarization creation and annihilation operators are

$$[a_{\mathbf{k}L}, a_{\mathbf{k}L}^{\dagger}] = 1, \qquad [a_{\mathbf{k}R}, a_{\mathbf{k}R}^{\dagger}] = 1. \tag{136}$$

So all that survives after using the commutation relations is

$$\mathbf{S} = \sum_{\mathbf{k}} \hbar\hat{k} \left[a_{\mathbf{k}L}^{\dagger} a_{\mathbf{k}L} - a_{\mathbf{k}R}^{\dagger} a_{\mathbf{k}R} \right] \tag{137}$$

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This apparently involves number operators for each circular polarization. The left states contribute $+\hbar\hat{k}$ and the right states contribute $-\hbar\hat{k}$ to the total angular momentum. So in a sense, one can consider that photons carry an intrinsic angular momentum of magnitude $\hbar$. Then they can be considered as particles with spin-1. The numbers operators in this expression count the number of photons in each helicity or circular polarization state.

The creation operators generate quanta of the EM field, in some sense. There are the linear polarization operators and the circular ones. We might think of the circular state as a linear combination of two linear states, and vice versa. Does a photon have an intrinsic angular momentum? Perhaps it is not a reasonable question. Depending on what is measured, you may only see either the linear components or only the circular components. The basis you would use is determined by how the experiment is performed, i.e., by what states your experiment projects out. However, some photon-atom interactions, for example, may be most sensitive to the circular components. Then that type of experiment would view the photons as having circular polarization and as being spin-1 particles.

2 Interactions of EM fields with matter

EM fields are always coupled to charged particles. Here coupling of the quantized EM field to nonrelativistic charges is considered. It is expected that this is a situation where a perturbation approach can be applied. It means, we think the eigenstates of some material problem are known, and we want to see the perturbations due to the application of light. Or, it could be we want to see the emission of light that is expected, either spontaneously due to fluctuations in the EM field, or stimulated by the EM field. This is a brief summary of how to go about these calculations.

The usual prescription to include interactions is to modify the four-momentum of the particle of charge q, according to the effects of the field it experiences,

$$\mathbf{p} \longrightarrow \mathbf{p} - \frac{q}{c}\mathbf{A}, \qquad E \longrightarrow E - qA_0 \tag{138}$$

where $\mathbf{A}$ is the vector potential and A_0 is the scalar potential ϕ. The energy operator is the usual $E = i\hbar\frac{\partial}{\partial t}$ and the momentum operator is the usual $\mathbf{p} = -i\hbar\nabla$. The original problem in the absence of the fields is a Schrödinger problem $H\Psi = E\Psi$,

$$\left[\frac{\mathbf{p^2}}{2m} + V(\mathbf{r})\right]\Psi(\mathbf{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) \tag{139}$$

When the EM fields are now turned on, this changes to

$$\left[\frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + V(\mathbf{r}) + qA_0\right]\Psi(\mathbf{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) \tag{140}$$

Then the effective Hamiltonian for the perturbed problem is just

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + V(\mathbf{r}) + qA_0 \tag{141}$$

The simplest case is to consider a single particle or charge as it interacts with the field. In addition, there is supposed to be the Hamiltonian for the free EM fields themselves, as we quantized in the previous section. So the total effective Hamiltonian for particle in the field is

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}) - \frac{q}{2mc}(\mathbf{A}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{A}) + \frac{q^2\mathbf{A^2}}{2mc^2} + qA_0 + H_{\text{photons}} \tag{142}$$

The first two terms are the free particle, the next three terms are the interaction with the EM fields, and the last term is the free photon Hamiltonian, whose eigenstates are states with certain numbers of photons are given wavevectors and polarizations.

Although not necessary, we can continue to use the radiation gauge, where $\nabla\cdot\mathbf{A} = \mathbf{0}$ and $A_0 = 0$, as we used for the free photons. This implies $\mathbf{p}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{p}$. Then there are only two interaction

terms, one that is linear in $\mathbf{A}$ and one that is quadratic in $\mathbf{A}$. The linear terms will involve individual photon creation and annihilation terms, corresponding to single-photon processes that can be analyzed in first order perturbation theory. The quadratic terms correspond to two-photon processes even in first order PT, and probably to get those correct would actually require second order PT. However, they should be smaller. Thus the leading perturbation is due to only the single-photon term,

$$H_1 = -\frac{q}{mc}\mathbf{A} \cdot \mathbf{p} + \frac{q^2\mathbf{A^2}}{2mc^2} \approx -\frac{q}{mc}\mathbf{A} \cdot \mathbf{p} \tag{143}$$

Note that the $\mathbf{A}$ operator is due to the EM field (creation/annihilation) while the $\mathbf{p}$ operator is associated with the charged particle momentum only. These would act on composite bra or kets involving both the EM field state and the particle state.

In a basic process where the final state has one less photon than the initial state, a photon was absorbed by the matter. On the other hand, if the final state has one more photon than the initial state, a photon was emitted by the matter.

In absorption, the absorbed photon could come from any direction $d\Omega$. In emission, the emitted photon can go out in any direction $d\Omega$. In either case, some density of states is involved in the calculation of the rate for the process, consistent with conservation of energy and other variables.

2.1 First order perturbations: single photon events

The amplitude for a transition between some initial and final states is given by a matrix element, which goes into Fermi's Golden Rule to get the transition rate:

$$w_{I \to F} = \frac{2\pi}{\hbar}\left|\langle \Psi_F|H_1|\Psi_I\rangle\right|^2 \rho \tag{144}$$

where ρ is some density of states for the photon involved. The interaction can be written using the quantized EM field $\mathbf{A}$,

$$H_1 = \frac{-q}{mc}\sqrt{\frac{2\pi c^2\hbar}{V}}\sum_{\mathbf{k}}\frac{1}{\sqrt{\omega_{\mathbf{k}}}}\left[\hat{\epsilon}_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{\epsilon}^{\dagger}_{\mathbf{k}\alpha}a^{\dagger}_{\mathbf{k}\alpha}e^{-i\mathbf{k}\cdot\mathbf{r}}\right] \cdot \mathbf{p} \tag{145}$$

So we need matrix elements of this, between states with given numbers of photons, and initial and final states of the matter. Especially, the quantum process of light emission or absorption requires use of the basic matrix elements of the creation/annhilation operators.

The states are products of a state for the particle and a state for the fields. So we write in general,

$$|\Psi\rangle = |\psi\rangle|\,(n)\rangle \tag{146}$$

where ψ refers to the particle state, and (n) is a shorthand notation for the occupation numbers of all of the photon modes. In absorption or emission, we suppose that only one of the photon modes changes its occupation number, because we are dealing with single-photon processes. Then the general matrix element for the process is

$$\langle \Psi_F|H_1|\Psi_I\rangle = \frac{-q}{m}\sqrt{\frac{h}{V}}\sum_{\mathbf{k}\alpha}\frac{1}{\sqrt{\omega_{\mathbf{k}}}}\quad \left\{\langle (n)_f|a_{\mathbf{k}\alpha}e^{-i\omega_{\mathbf{k}}t}|(n)_i\rangle\langle\psi_f|e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\epsilon}_{\mathbf{k}\alpha}\cdot\mathbf{p}|\psi_i\rangle\right.$$

$$\left. + \langle (n)_f|a^{\dagger}_{\mathbf{k}\alpha}e^{i\omega_{\mathbf{k}}t}|(n)_i\rangle\langle\psi_f|e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{\epsilon}_{\mathbf{k}\alpha}\cdot\mathbf{p}|\psi_i\rangle\right\} \tag{147}$$

In first order time-dependent perturbation theory, if an effective potential that is acting is $V(t) = V_o e^{-i\omega t} + V_o^{\dagger}e^{+i\omega t}$, the transition amplitude according to the Born approximation is a matrix element,

$$\langle f|U(t,0)|i\rangle = \frac{-i}{\hbar}\int_0^t d\tau\; e^{i\omega_{fi}\tau}\langle f|V(\tau)|i\rangle, \quad \omega_{fi} \equiv \frac{(E_f - E_i)}{\hbar} \tag{148}$$

That's the same form we have for this problem. The frequency here depends on the difference in energy of the initial and final states of the matter, which is being perturbed by the radiation.

Now when combined with the frequency of the radiation field (for one particular mode), the time integrations needed are simple, one for absorption or destruction of a photon, and one for emission or creation of a photon,

$$\int_0^t d\tau \ e^{i\omega_{fi}\tau}e^{-i\omega_{\mathbf{k}}\tau} = \frac{e^{i(\omega_{fi}-\omega_{\mathbf{k}})t} - 1}{i(\omega_{fi} - \omega_{\mathbf{k}})}, \quad \text{for absorption} \tag{149}$$

$$\int_0^t d\tau \ e^{i\omega_{fi}\tau}e^{+i\omega_{\mathbf{k}}\tau} = \frac{e^{i(\omega_{fi}+\omega_{\mathbf{k}})t} - 1}{i(\omega_{fi} + \omega_{\mathbf{k}})}, \quad \text{for emission} \tag{150}$$

These expressions tend to get large where the denominators go to zero. Since this occurs at different places, we can consider absorption and emission separately.

For Absorption: The squared matrix element is the probability to find the system in the desired final state after a time t has passed. When we let $t \to \infty$, and divide by the time, it gives the **transition rate**. This is one thing, for example, that can be calculated. In the problem of finding the dielectric function of a material, however, we will want to find some different quantities. Thus let's just summarize this briefly.

For simplicity, supposed there is only one wavelength of light present in the radiation field, the one that is exactly tuned to be absorbed! The radiation matrix element that is needed is

$$\langle (n)_f | a_{\mathbf{k}\alpha} | (n)_i \rangle = \sqrt{n_{\mathbf{k}\alpha}} \tag{151}$$

This is an extreme approximation, because the light should really have some spectral distribution. This can be corrected later. Then the probability for absorbing this one mode is the squared matrix element (squared transition amplitude),

$$|\langle f|U|i\rangle|^2 = \frac{2\pi q^2 n_{\mathbf{k}\alpha}}{m^2 V \hbar \omega_{\mathbf{k}}} \left| \langle \psi_f | e^{i\mathbf{k}\cdot\mathbf{r}} \hat{\epsilon}_{\mathbf{k}\alpha} \cdot \mathbf{p} | \psi_i \rangle \right|^2 \left[\frac{\sin \frac{(\omega_{fi}-\omega_{\mathbf{k}})}{2}t}{\frac{(\omega_{fi}-\omega_{\mathbf{k}})}{2}} \right]^2 \quad \text{(absorption)} \tag{152}$$

In the limit of large time, the last factor acts like a delta functon in its argument in the sine. So this result is particularly peaked at the resonance, $\omega_{\mathbf{k}} = \omega_{fi}$, which just displays the conservation of energy in the absorption process.

Now consider a certain spectrum of radiation, but all travelling in one direction (a beam). The sum over wave vector can be considered in this situation as an integral over only the magnitude of $\mathbf{k}$, or equivalently, over the frequency $\omega_{\mathbf{k}}$. The beam is characterized by the intensity of its light at different frequencies, i.e., its spectrum, which could be very narrow or very wide. Let a function $I_\alpha(\omega)$ describe the intensity of light per unit frequency interval. Then $I_\alpha(\omega)d\omega$ is the energy per time per area in the beam, for modes of frequency ω and polarization α. Also, intensity in waves is their energy density times their speed. For a collection of N monochromatic photons, the intensity could be written

$$I = \frac{N}{V}\hbar\omega c, \implies N = V\frac{I}{\hbar\omega c} \tag{153}$$

Here we have a sum and not exactly monochromatic light. If we change a sum over modes into a continuous integral, we usually do the replacement,

$$\sum_{\mathbf{k}} \longrightarrow \frac{V}{(2\pi)^3} \int d^3\mathbf{k} \tag{154}$$

But that would apply to photons travelling in all possible directions. If we instead just sum up the photons, each of energy $\hbar\omega$ travelling at the speed c along the chosen beam direction, the number of photons (in volume V in the beam) will be described by

$$\sum_{\mathbf{k}} n_{\mathbf{k}\alpha} \longrightarrow V \int d\omega \ \frac{I_\alpha(\omega)}{\hbar\omega c} \tag{155}$$

The absorption transition rate involves the limit,

$$w = \lim_{t \to \infty} \frac{1}{t} \sum_{\mathbf{k}\alpha} |\langle f|U|i\rangle|^2 \tag{156}$$

For the matter in a beam of radiation, this leads to

$$w = \lim_{t \to \infty} \frac{1}{t} V \int d\omega \, \frac{I_\alpha(\omega)}{\hbar \omega c} \frac{2\pi q^2}{m^2 V \hbar \omega} \left| \langle \psi_f | e^{i\mathbf{k}\cdot\mathbf{r}} \hat{\epsilon}_{\mathbf{k}\alpha} \cdot \mathbf{p} | \psi_i \rangle \right|^2 \left[\frac{\sin \frac{(\omega_{fi}-\omega)}{2} t}{\frac{(\omega_{fi}-\omega)}{2}} \right]^2 \tag{157}$$

When the limit is taken, the last function on the end produces a delta function enforcing $\omega = \omega_{fi}$. Using $x = \frac{(\omega_{fi}-\omega)}{2} t$, the weight is found to be

$$\frac{1}{t} \int_{-\infty}^{+\infty} d\omega \left[\frac{\sin \frac{(\omega_{fi}-\omega)}{2} t}{\frac{(\omega_{fi}-\omega)}{2}} \right]^2 = \int_{-\infty}^{+\infty} dx \, 2 \left(\frac{\sin x}{x} \right)^2 = 2\pi \tag{158}$$

Therefore the transition rate for absorption of one polarization from the beam can be expressed

$$w = \frac{1}{c} \left(\frac{2\pi q}{m\hbar\omega_{fi}} \right)^2 I_\alpha(\omega_{fi}) \left| \langle \psi_f | e^{i\mathbf{k}\cdot\mathbf{r}} \hat{\epsilon}_{\mathbf{k}\alpha} \cdot \mathbf{p} | \psi_i \rangle \right|^2 \quad \text{(absorption)} \tag{159}$$

Emission: The radiation matrix element squared now is $n_{\mathbf{k}\alpha} + 1$. The first term corresponds to stimulated emission, caused by radiation already present, and the second is the spontaneious emission. The calculation of the stimulated rate follows the same algebra as for absorption. One will arrive at the rate in a beam to be

$$w = \frac{1}{c} \left(\frac{2\pi q}{m\hbar\omega_{if}} \right)^2 I_\alpha(\omega_{if}) \left| \langle \psi_f | e^{-i\mathbf{k}\cdot\mathbf{r}} \hat{\epsilon}_{\mathbf{k}\alpha} \cdot \mathbf{p} | \psi_i \rangle \right|^2 \quad \text{(stimulated emission)} \tag{160}$$

So this is really now much different from the absorption expression. Physically, however, it is completely different, especialy, in that the rocess will produce a new photon that is coherent with the present photons. This leads to masers, lasers, etc. The produced photon is just a copy of one of the original ones, and is also travelling in the same direction.

For now I don't discuss spontaneous emission, although it is a very interesting subject!

3 Electric polarization and dielectrics

Here I want to consider the basic theory for induced electric dipoles in matter, and how that leads to the electric permitivity $\epsilon(\omega)$. In optical systems, it is clear that the response of a medium to the radiation fields, i.e., photons, induces electric dipoles, and those in turn could react back on the radiation. The theory is related to that just discussed for using time-dependent perturbation theory applied to absorption and emission of photons.

An optical medium has a dielectric response due primarily to its electrons of charge $q = -e$, and their dipole moments induced by applied fields. I will consider this as a quantum problem, because I will include the effects of both the electric and magnetic fields, especially, what happens when a DC magnetic field is applied (Faraday effect).

Start from the simplest problem, the Drude model, where the optical medium is composed just from a gas of free (noninteracting) electrons, moving between fixed nuclei. Classically the problem is quite simple: the electric field in the radiation field at frequency ω displaces the electrons from their original positions, at the frequency of the radiation. Then it is easy to find the induced dipoles and do the necessary electrodynamics to get $\epsilon(\omega)$. The only possible difficulty: the gas oscillates as a whole, leading to plasma oscillations. But we aren't really considering this kind of collective mode, only the averaged response of individual charges. An individual charge follows Newton's Law, in the net field surrounding it, $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$,

$$m\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t}, \implies \mathbf{r}(t) = \frac{-q\mathbf{E}_0}{m\omega^2} e^{-i\omega t} = \frac{-q}{m\omega^2} \mathbf{E} \tag{161}$$

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The charge oscillates at the same frequency as the radiation, and its displacements of amplitude $-q\mathbf{E}_0/m\omega^2$ about its original position are proportional to the strength of the radiation. The charge's induced electric dipole moment is $\mathbf{d} = q\mathbf{r}$. If there are N charges in a volume V, or a volume density of $n = N/V$, then the net dipole moment per unit volume is the electric polarization,

$$\mathbf{P} = n\mathbf{d} = nq\mathbf{r} = -\frac{nq^2}{m\omega^2}\mathbf{E} \tag{162}$$

The total electric displacement can then be found (CGS units) to get the dielectric function,

$$\mathbf{D} = \epsilon\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P} = \left(1 - 4\pi\frac{nq^2}{m\omega^2}\right)\mathbf{E} \implies \epsilon(\omega) = 1 - 4\pi\frac{nq^2}{m\omega^2} \tag{163}$$

One can see the large oscillations and response will occur if $\epsilon \to 0$, which takes place at the plasma frequency,

$$\omega_p = \sqrt{\frac{4\pi nq^2}{m}} \tag{164}$$

Then it is usual to write the dielectric function for this simplest case as

$$\epsilon(\omega) = 1 - 4\pi\frac{nq^2}{m\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \tag{165}$$

Note that to convert the results to SI units, just recall that the charge must be re-scaled by the relation

$$q_{\text{CGS}}^2 \to \frac{q_{\text{SI}}^2}{4\pi\epsilon_0}; \qquad \omega_p = \sqrt{\frac{nq^2}{m\epsilon_0}} \quad \text{(SI units)} \tag{166}$$

where $\epsilon_0 = 8.854$ pF/m is the permitivity of vacuum. Furthermore, one can consider partially bound electrons with damped motion–it is a slight modification of the above and removes the infinite divergence at the plasma frequency.

What about the QM problem for the Drude model? The free electrons can be assumed to be in eigenstates of momentum, i.e, their unperturbed Hamiltonian only has kinetic energy:

$$H_{\text{e}} = \frac{\mathbf{p^2}}{2m} \tag{167}$$

Take the eigenstates as normalized plane waves:

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{V}}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{168}$$

First I'll consider a zero temperature problem. Just consider the effects on an individual electron and its interaction with the quantized radiation fields in the Coulomb gauge. Take the interaction due to single-photon processes, as discussed in the previous section:

$$H_1 = -\frac{q}{mc}\mathbf{A}\cdot\mathbf{p} \tag{169}$$

The perturbation should cause the electron to make transitions between the plane wave states. Let's see what happens. We know the expression for the radiation vector potential, so

$$H_1 = \frac{-q}{m}\sqrt{\frac{h}{V}}\sum_{\mathbf{k}}\frac{1}{\sqrt{\omega_{\mathbf{k}}}}\left[\hat{\epsilon}_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{\epsilon}_{\mathbf{k}\alpha}^{\dagger}a_{\mathbf{k}\alpha}^{\dagger}e^{-i\mathbf{k}\cdot\mathbf{r}}\right]\cdot\mathbf{p} \tag{170}$$

Now we don't necessarily need to know transition rates for this problem, we only want to know the expectation value of the electric dipole moment operator, $\mathbf{d} = q\mathbf{r}$. Unfortunately, we see right away, that for whatever state of the electron we might pick, this expectation will be zero, because of the photon operators. If the photon number does not change, the photon matrix elements will give zero.

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Thus I suspect we need to look at this problem with second order perturbation theory. Further, we need to be cleaer about what is the actual state of the photons being considered. However, that could be fairly simple, say, a state with n identical photons (a very large number) at the frequency for which we need to know $\epsilon(\omega)$. Even for this state, first order PT will not give an induced dipole moment.

It's possible that states of the time-dependent Hamiltonian, $H_e + H_1$ can be solved without a perturbation approach. Try that first.