Name Electro Dynamic

Instructions: Use SI units. Short answers! No derivations here, just state your responses clearly.

1. (2) Write an integral expression for the magnetic induction generated in unbounded vacuum by a current density $\vec{J}(\vec{r})$.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r\ \vec{J}(\vec{r}') \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}.$$

2. (2) Now give the corresponding differential equation which the same magnetic induction must satisfy.

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r}).$$

3. (2) What vector potential $\vec{A}(\vec{r})$ is generated by the same current distribution $\vec{J}(\vec{r})$?

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r\ \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}.$$

This is for the Coulomb gauge, where $\vec{\nabla}\cdot\vec{A}=0$.

4. (4) How do the scalar and vector potentials determine the fields for a time-dependent problem.

They give the electric and magnetic fields by

$$\vec{B} = \vec{\nabla} \times \vec{A}, \qquad \vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}.$$

5. (2) Write out the integral form of Ampere's Law.

$$\oint \vec{H}\cdot d\vec{l} = I_{\text{enclosed}}.$$

6. (3) Give a formula for the potential energy of a magnetic dipole when placed in an external magnetic induction $\vec{B}$.

$$U = -\vec{m}\cdot\vec{B}.$$

7. (4) State the boundary conditions on $\vec{B}$ and $\vec{H}$ at an interface between two linear nonconducting permeable media.

$$\vec{B}_1\cdot\hat{n} = \vec{B}_2\cdot\hat{n}, \qquad \vec{H}_1\times\hat{n} = \vec{H}_2\times\hat{n}.$$

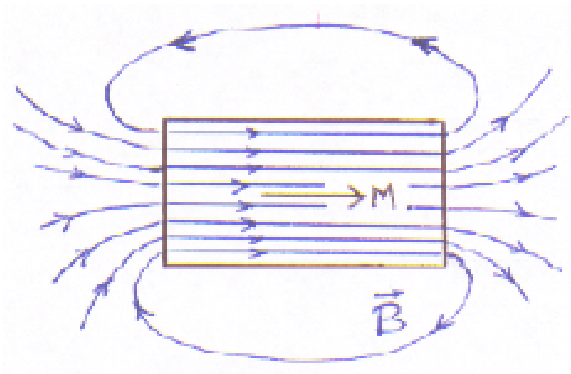
Here $\hat{n}$ is a unit vector normal to the boundary.

8. (2) Give the relation defining magnetic field $\vec{H}$ in terms of magnetic induction $\vec{B}$ and magnetization $\vec{M}$.

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}.$$

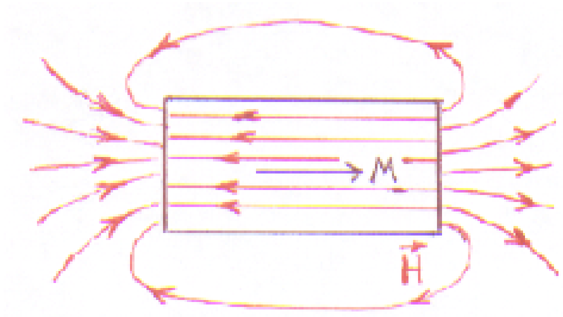
9. (2) A cylindrical permanent magnet has a uniform magnetization M along its long axis. Sketch the lines of its $\vec{B}$-field both inside and outside.

(See Jackson, Fig. 5.11, for example.)



10. (2) Sketch the lines of its $\vec{H}$-field for the same permanent magnet.

(See Jackson, Fig. 5.11, for example.)



11. (3) Give an expression for the magnetic dipole moment of a current distribution $\vec{J}(\vec{r})$:

$$\vec{m} = \frac{1}{2}\int d^3r\ \vec{r} \times \vec{J}(\vec{r}).$$

12. (4) Write out Maxwell's equations for harmonic time-dependent fields:

$$\vec{\nabla}\cdot\vec{B} = 0, \qquad \vec{\nabla}\times\vec{E} - i\omega\vec{B} = 0,$$
$$\vec{\nabla}\cdot\vec{D} = \rho, \qquad \vec{\nabla}\times\vec{H} + i\omega\vec{D} = \vec{J}.$$

13. (3) What quantity describes the flux of energy carried in an electromagnetic wave? Give the name and defining formula.

The Poynting vector, defined by the formula,

$$\vec{S} = \vec{E}\times\vec{H}.$$

14. (2) How does the following quantity transform under spatial inversion?

$$\vec{L}_{\rm em} = \frac{1}{c^2}\int d^3r\ \vec{r}\times(\vec{E}\times\vec{H})$$

Since $\vec{E}(-\vec{r}) = -\vec{E}(\vec{r})$ (odd), and $\vec{H}(-\vec{r}) = \vec{H}(\vec{r})$ (even), while $\vec{r}$ itself reverses (odd), $\vec{L}_{\rm em}$ is EVEN under spatial inversion.

15. (2) How does $\vec{L}_{\rm em}$ transform under time reversal?

Since $\vec{E}$ even while $\vec{H}$ is odd, their product is ODD under time reversal. These are the usual properties for angular momentum.

16. (2) A plane electromagnetic wave at angular frequency ω propagates in the z-direction in a medium with permittivity ϵ and permeability μ. Write an expression for its wavevector $\vec{k}$.

$$\vec{k} = \omega\sqrt{\epsilon\mu}\ \hat{z}.$$

17. (3) For the same plane wave, write an expression for its electric field $\vec{E}(\vec{r}, t)$.

$$\vec{E}(\vec{r}, t) = E_0\, \hat{\varepsilon}\ \exp\{i(\vec{k} \cdot \vec{r} - \omega t)\},$$

where polarization unit vector $\hat{\varepsilon}$ is perpendicular to $\vec{k}$.

18. (3) Also for the same plane wave, what is the associated magnetic induction $\vec{B}(\vec{r}, t)$?

$$\vec{B}(\vec{r}, t) = E_0\, \frac{\vec{k}}{\omega} \times \hat{\varepsilon}\ \exp\{i(\vec{k} \cdot \vec{r} - \omega t)\},$$

as you can obtain quickly from the harmonic form of Farady's Law.

19. (2) Give a relation between the intensity I_0 in a plane EM wave and its time-averaged energy density u.

$$I_0 = \frac{1}{2}\mathrm{Re}\left\{\vec{E} \times \vec{H}^*\right\} = \frac{1}{\sqrt{\epsilon\mu}} u, \qquad u = \frac{1}{2}\mathrm{Re}\left\{\frac{\epsilon}{2}\vec{E} \cdot \vec{E}^* + \frac{1}{2\mu}\vec{B} \cdot \vec{B}^*\right\}.$$

20. (2) When light undergoes total internal reflection at an interface between two optical media with indexes n (incident side) and n' (refraction side), how large is the incident angle θ?

The incident angle must be greater than the critical angle, defined by $\theta_c = \sin^{-1}(n'/n)$, requiring $n' < n$.

21. (3) For the same interface, at what incident angle will the reflected wave be totally polarized? And in which direction is it polarized?

The incident angle must be the Brewster angle, $\theta_B = \tan^{-1}(n'/n)$, at which there is no reflection for $\vec{E}$ polarized within the plane of incidence. Then for this incident angle the reflected light is 100% polarized perpendicular to the plane of incidence.

22. (2) A dispersive medium has a frequency-dependent dielectric function $\epsilon(\omega)$ (dispersion). For an EM wave-packet of narrow bandwidth passing through this medium, describe at least one important effect caused by the dispersion.

1) Different wavevector components travel at different speeds, leading to temporal spreading.

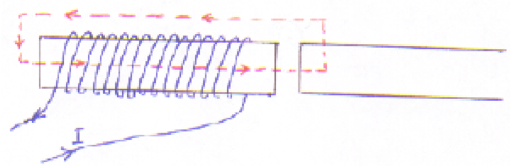
2) Different wavevector components suffer a range of refraction angles, leading to spatial spreading.

3) The energy flows at a velocity different from the phase velocity.

Name____________________________________

Instructions: Use SI units. Please Write your derivations and final answers on these pages. Explain your reasoning for full credit. One-page note summary is allowed.

23. (16) An electromagnet is made by winding a coil with $N = 2000$ turns on a cylindrical piece of soft iron with length $l = 4.0$ cm and radius $a = 4.0$ mm, with high permeability $\mu = 400\mu_0$. One end of the electromagnet is placed a distance $d = 1.0$ mm from a long rod of the same cross section, with extremely high permeability (take $\mu' = \infty$).



Assume that the fields $\vec{B}_{\rm iron}$ and $\vec{H}_{\rm iron}$ within the iron are nearly uniform, and that $d \ll a \ll l$. The arrangement is surrounded by vacuum. A constant current $I = 5.0$ A is flowing in the coil.

a) (8) Apply Ampere's Law, showing the path you use, and estimate the magnitudes $B_{\rm iron}$ and $H_{\rm iron}$, and the average magnetization $M_{\rm iron}$. Give them with correct SI units.

The path for Ampere's Law is indicated in the Figure. Continuity of $\vec{B}_n$ gives $B_{\rm iron} = B_{\rm gap} = B_{\rm rod}$, hence, $H_{\rm rod} = 0$, and $\mu H_{\rm iron} = \mu_0 H_{\rm gap}$. Then Ampere's Law is

$$H_{\rm iron} l + H_{\rm gap} d = NI, \quad H_{\rm iron} = \frac{NI}{l + \frac{\mu}{\mu_0} d} = \frac{2000 \times 5}{0.04 + 400 \times 0.001} = 22700 \text{ A/m}.$$

Then the magnetic induction is

$$B_{\rm iron} = \mu H_{\rm iron} = \frac{NI}{\frac{l}{\mu} + \frac{d}{\mu_0}} = 400\mu_0 H_{\rm iron} = 400 \times 4\pi \times 10^{-7} \times 22700 = 11.4\text{T}.$$

The associated magnetization comes from the basic definition:

$$B_{\rm iron} = \mu H_{\rm iron} = \mu_0(H_{\rm iron} + M_{\rm iron}),$$

$$M_{\rm iron} = \left(\frac{\mu}{\mu_0} - 1\right) H_{\rm iron} = (400 - 1) \times 22700 = 9.07 \times 10^6 \text{A/m}.$$

b) (4) Determine the associated values of $B_{\rm gap}$ and $H_{\rm gap}$ in the gap between the electromagnet and the other rod.

From the normal boundary conditions

$$B_{\rm gap} = B_{\rm iron} = 11.4\text{T}, \qquad H_{\rm gap} = \frac{B_{\rm gap}}{\mu_0} = \frac{B_{\rm iron}}{\mu_0} = \frac{\mu}{\mu_0} H_{\rm iron}.$$

$$H_{\rm gap} = 400 H_{\rm iron} = 400 \times 22700 = 9.09 \times 10^6 \text{A/m}.$$

c) (4) Estimate the force of attraction between the electromagnet and the rod, in Newtons. Does the result depend on d?

The magnetic induction in the gap acts on the effective surface magnetic charges on the end of the electromagnet. The magnetic energy density gives the force per unit area, therefore,

$$F = u_{\rm gap} A = \frac{B_{\rm gap}^2}{2\mu_0} \pi a^2 = \frac{11.4^2}{2 \times 4\pi \times 10^{-7}} \pi (0.004)^2 = 2610 \text{ N}.$$

You can also get this from energy analysis, but that requires careful consideration of the work done by the current source. The result does depend on d, via the dependence of $B_{\rm gap}$ on d.

24. (12) Consider an EM plane wave propagating within a crystal of permittivity $\epsilon = 2.25\,\epsilon_0 = (9/4)\,\epsilon_0$ (for example, inside a diamond crystal), which is incident from inside on some face of the crystal. The crystal is surrounded by vacuum. Assume $\mu = \mu_0$ everywhere.

a) (8) Consider an incident angle $\theta = 60°$, which results in TIR. Take the x-axis along the boundary and the z-axis pointing out of the crystal, perpendicular to the boundary. If the incident electric field is

$$\vec{E}(\vec{r},t) = E_0\ \hat{y}\ \exp\{i[\vec{k}\cdot\vec{r} - \omega t]\} = E_0\ \hat{y}\ \exp\{i[k(\sin\theta\ x + \cos\theta\ z) - \omega t]\},$$

write an expression for the evanescent field $\vec{E}\,'(x,z,t)$ on the vacuum side, explicitely displaying its dependence on x and z, and correct amplitude.

The wave on the refraction side has a similar expression and is also polarized along y, but by Snell's Law its angle of refraction will be complex:

$$\sin\theta' = \frac{n}{n'}\sin\theta = \frac{\sqrt{\epsilon}}{\sqrt{\epsilon_0}}\sin 60° = \frac{3}{2}\,\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4},$$

$$\cos\theta' = \sqrt{1-\sin^2\theta'} = \sqrt{1-\frac{27}{16}} = i\,\frac{\sqrt{11}}{4}$$

Using $k' = n'\omega/c = \omega/c$, the wavevector components are

$$k'\sin\theta' = \frac{\omega}{c}\,\frac{3\sqrt{3}}{4}, \quad k'\cos\theta' = \frac{\omega}{c}\,\frac{i\sqrt{11}}{4}.$$

Furthermore, its amplitude is obtained from a Fresnel formula:

$$\frac{E_0'}{E_0} = \frac{2\sqrt{\epsilon}\cos\theta}{\sqrt{\epsilon}\cos\theta + \sqrt{\epsilon'}\cos\theta'} = \frac{2\sqrt{9/4}\ (1/2)}{\sqrt{9/4}\ (1/2) + 1\ i(\sqrt{11}/4)} = \frac{1}{10}(9 - i\ \sqrt{99}).$$

Then the evanescent field is

$$\vec{E}'(\vec{r},t) = E_0'\ \hat{y}\ \exp\{i[k'(\sin\theta'\ x + \cos\theta'\ z) - \omega t]\}$$

$$\vec{E}'(\vec{r},t) = \frac{9 - i\sqrt{99}}{10}\ E_0\ \hat{y}\ \exp\left\{i\frac{\omega}{c}\left[\frac{3\sqrt{3}}{4}x + i\frac{\sqrt{11}}{4}z - ct\right]\right\}$$

b) (4) Over what distance into the vacuum (measured in free space wavelengths) does the evanescent wave decay by a factor of e^{-1} in amplitude?

By inspection of the z-dependent, which is a decaying exponential (exactly what it means by evanescent wave!), we see the dependence,

$$E' \propto \exp\left\{-\frac{\sqrt{11}}{4}\frac{\omega}{c}z\right\} = \exp\{-z/\delta\},$$

where

$$\delta = \frac{4}{\sqrt{11}}\frac{c}{\omega} = \frac{4}{\sqrt{11}}\frac{\lambda}{2\pi} \approx 0.19\lambda.$$

25. (16) EM waves in a plasma interact with the dielectric function,

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}.$$

a) (4) Assuming $\mu = \mu_0$, derive the dispersion relation giving $\omega(k)$ for EM waves in a plasma.

Use the basic definition for plane waves:

$$k = \omega\sqrt{\epsilon\mu} = \omega\sqrt{\epsilon_0\mu_0\left(1 - \omega_p^2/\omega^2\right)} = \sqrt{\frac{1}{c^2}\left(\omega^2 - \omega_p^2\right)},$$

$$\omega^2 = \omega_p^2 + c^2k^2.$$

b) (4) Based on your $\omega(k)$ [or $k(\omega)$], what happens to waves with $\omega < \omega_p$, that enter this medium?

$\omega < \omega_p$ requires an imaginary value of k. But if k is imaginary, this will lead to a nonpropagating evanescent wave (decaying exponential). The waves with $\omega < \omega_p$ generally, then, are reflected from the plasma, very similar to the total internal reflection effect.

c) (4) Now suppose the plasma has $NZ = 1.0 \times 10^{22}$ electrons/m^3, and consider waves with $\omega = 3\omega_p$. What is the phase velocity v_p of the waves?

Use the basic definitions and results from above:

$$k = \sqrt{\frac{1}{c^2}\left[(3\omega_p)^2 - \omega_p^2\right]} = \sqrt{8}\,\frac{\omega_p}{c}.$$

$$v_p = \frac{\omega}{k} = \frac{3\omega_p}{\sqrt{8}\,\omega_p/c} = \frac{3}{\sqrt{8}}c \approx 1.06c.$$

d) (4) For the same parameters as (c), what is the group velocity, $v_g = \frac{d\omega}{dk}$?

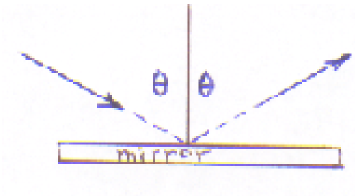
Doing the required calculus:

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(\omega_p^2 + c^2k^2)^{1/2} = \frac{c^2k}{\sqrt{\omega_p^2 + c^2k^2}} = \frac{c^2k}{\omega} = \frac{c^2}{v_p}$$

$$v_g = \frac{c^2}{3c/\sqrt{8}} = \frac{\sqrt{8}}{3}c \approx 0.94c.$$

So the group velocity is reasonable, while the phase velocity exceeds the speed of light in vacuum.

26. (10) A plane wave of intensity $I_0 = 20.0$ kW/cm^2 is incident on a perfectly reflecting mirror at angle of incidence $\theta = 60°$. Determine the radiation pressure on the mirror in N/m^2.



The time-averaged momentum density in the field in a vacuum is given by

$$\vec{g}_{\rm ave} = \frac{1}{2}\mathrm{Re}\left\{\frac{1}{c^2}\vec{E}\times\vec{H}^*\right\}$$

Multiplication by c converts this to a flux of momentum. But only the normal component, proportional to $\cos\theta$, changes on reflection. Also, the momentum is spread out over an area increasing as $1/\cos\theta$. The pressure is the change in momentum per time per area on the mirror,

$$P = c\frac{g_{\rm inc.\ z} - g_{\rm ref.\ z}}{1/\cos\theta} = 2cg_{\rm ave}\cos^2\theta.$$

This can be rewritten using the intensity $I_0 = \frac{1}{2}|\vec{E}\times\vec{H}^*|$, then $|\vec{g}_{\rm ave}| = I_0/c^2$, and

$$P = 2\frac{I_0}{c}\cos^2\theta = 2\ \frac{20000/(0.01)^2}{3.0\times 10^8}\left(\frac{1}{2}\right)^2 = 0.33\ \mathrm{N/m^2}.$$

27. (10) The imaginary part of a dielectric function is known to be

$$\frac{\epsilon_I(\omega)}{\epsilon_0} = \frac{\gamma\omega}{\omega^2+\gamma^2}.$$

a) (4) Apply the Kramers-Kronig relations to obtain the real part of $\epsilon(\omega)$.

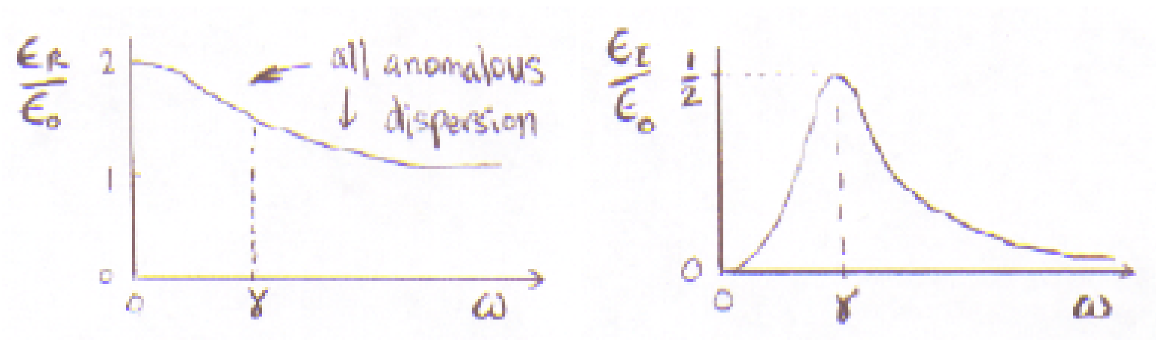
Use the simplest form, with the principal valued integral, closing the contour in the upper half plane:

$$\frac{\epsilon_R(\omega)}{\epsilon_0} - 1 = \frac{1}{\pi}P\int_{-\infty}^{\infty} dz\, \frac{\epsilon_I(z)}{z-\omega} = \frac{1}{\pi}P\int_{-\infty}^{\infty} dz\, \frac{1}{z-\omega}\,\frac{\gamma z}{z^2+\gamma^2}$$

$$= \frac{1}{\pi}\left[\pi i \text{ res}(\omega) + 2\pi i \text{ res}(i\gamma)\right] = i\left[\frac{\gamma\omega}{\omega^2+\gamma^2} + \frac{2\gamma\, i\gamma}{(i\gamma-\omega)(2i\gamma)}\right] = \frac{\gamma^2}{\omega^2+\gamma^2}$$

$$\frac{\epsilon_R(\omega)}{\epsilon_0} = 1 + \frac{\gamma^2}{\omega^2+\gamma^2}$$

b) (4) Sketch the real and imaginary parts of $\epsilon(\omega)$ vs. ω, and identify the regions of normal and anomalous dispersion.



c) (2) Determine the complex function $\epsilon(\omega)$ and locate its poles in the complex ω-plane. Do its poles occur in the region associated with causal response?

Adding the real and imaginary parts give easily

$$\frac{\epsilon(\omega)}{\epsilon_0} = \frac{\epsilon_R(\omega) + i\,\epsilon_I(\omega)}{\epsilon_0} = 1 + \frac{\gamma^2}{\omega^2+\gamma^2} + i\,\frac{\gamma\omega}{\omega^2+\gamma^2}$$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{i\gamma(\omega - i\gamma)}{\omega^2+\gamma^2} = 1 + \frac{i\gamma}{\omega+i\gamma}.$$

The Only pole is at $\omega = -i\gamma$, in the lower half plane. Thus, $\epsilon(\omega)$ is analytic in the upper half plane, as required by causality.