conjugation which does not affect the question of orthonormality), the result we already have for the columns of a unitary matrix tells us the rows of U are orthonormal.

Proof 2. Since $U^\dagger U = I$,

$$\begin{aligned}\delta_{ij} &= \langle i|I|j\rangle = \langle i|U^\dagger U|j\rangle \\ &= \sum_k \langle i|U^\dagger|k\rangle\langle k|U|j\rangle \\ &= \sum_k U^\dagger_{ik}U_{kj} = \sum_k U^*_{ki}U_{kj}\end{aligned} \tag{1.6.22}$$

which proves the theorem for the columns. A similar result for the rows follows if we start with the equation $UU^\dagger = I$. Q.E.D.

Note that $U^\dagger U = I$ and $UU^\dagger = I$ are not independent conditions.

*Exercise 1.6.4.** It is assumed that you know (1) what a *determinant* is, (2) that det $\Omega^T =$ det Ω (T denotes transpose), (3) that the determinant of a product of matrices is the product of the determinants. [If you do not, verify these properties for a two-dimensional case

$$\Omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with det $\Omega = (\alpha\delta - \beta\gamma)$.] Prove that the determinant of a unitary matrix is a complex number of unit modulus.

*Exercise 1.6.5.** Verify that $R(\frac{1}{2}\pi\mathbf{i})$ is unitary (orthogonal) by examining its matrix.

Exercise 1.6.6. Verify that the following matrices are unitary:

$$\frac{1}{2^{1/2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \qquad \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

Verify that the determinant is of the form $e^{i\theta}$ in each case. Are any of the above matrices Hermitian?

1.7. Active and Passive Transformations

Suppose we subject all the vectors $|V\rangle$ in a space to a unitary transformation

$$|V\rangle \to U|V\rangle \tag{1.7.1}$$

Under this transformation, the matrix elements of any operator Ω are modified as follows:

$$\langle V'|\Omega|V\rangle \to \langle UV'|\Omega|UV\rangle = \langle V'|U^\dagger \Omega U|V\rangle \tag{1.7.2}$$

It is clear that the same change would be effected if we left the vectors alone and subjected all operators to the change

$$\Omega \to U^{\dagger}\Omega U \tag{1.7.3}$$

The first case is called an *active transformation* and the second a *passive transformation*. The present nomenclature is in reference to the vectors: they are affected in an active transformation and left alone in the passive case. The situation is exactly the opposite from the point of view of the operators.

Later we will see that the physics in quantum theory lies in the matrix elements of operators, and that active and passive transformations provide us with two equivalent ways of describing the same physical transformation.

*Exercise 1.7.1.** The *trace* of a matrix is defined to be the sum of its diagonal matrix elements

$$\mathrm{Tr}\,\Omega = \sum_i \Omega_{ii}$$

Show that

(1) $\mathrm{Tr}(\Omega\Lambda) = \mathrm{Tr}(\Lambda\Omega)$
(2) $\mathrm{Tr}(\Omega\Lambda\theta) = \mathrm{Tr}(\Lambda\theta\Omega) = \mathrm{Tr}(\theta\Omega\Lambda)$ (The permutations are *cyclic*).
(3) The trace of an operator is unaffected by a unitary change of basis $|i\rangle \to U|i\rangle$. [Equivalently, show $\mathrm{Tr}\,\Omega = \mathrm{Tr}(U^{\dagger}\Omega U)$.]

Exercise 1.7.2. Show that the determinant of a matrix is unaffected by a unitary change of basis. [Equivalently show $\det \Omega = \det(U^{\dagger}\Omega U)$.]

1.8. The Eigenvalue Problem

Consider some linear operator Ω acting on an arbitrary *nonzero* ket $|V\rangle$:

$$\Omega|V\rangle = |V'\rangle \tag{1.8.1}$$

Unless the operator happens to be a trivial one, such as the identity or its multiple, the ket will suffer a nontrivial change, i.e., $|V'\rangle$ will not be simply related to $|V\rangle$. So much for an arbitrary ket. Each operator, however, has certain kets of its own, called its *eigenkets*, on which its action is simply that of rescaling:

$$\Omega|V\rangle = \omega|V\rangle \tag{1.8.2}$$

Equation (1.8.2) is an eigenvalue equation: $|V\rangle$ is an *eigenket* of Ω with *eigenvalue* ω. In this chapter we will see how, given an operator Ω, one can systematically determine all its eigenvalues and eigenvectors. How such an equation enters physics will be illustrated by a few examples from mechanics at the end of this section, and once we get to quantum mechanics proper, it will be eigen, eigen, eigen all the way.