operations can be carried out. We shall now see that in the same manner a linear operator can be represented in a basis by a set of n^2 numbers, written as an $n \times n$ matrix, and called its *matrix elements* in that basis. Although the matrix elements, just like the vector components, are basis dependent, they facilitate the computation of all basis-independent quantities, by rendering the abstract operator more tangible.

Our starting point is the observation made earlier, that the action of a linear operator is fully specified by its action on the basis vectors. If the basis vectors suffer a change

$$\Omega|i\rangle = |i'\rangle$$

(where $|i'\rangle$ is known), then any vector in this space undergoes a change that is readily calculable:

$$\Omega|V\rangle = \Omega \sum_i v_i|i\rangle = \sum_i v_i \Omega|i\rangle = \sum_i v_i|i'\rangle$$

When we say $|i'\rangle$ is known, we mean that its components in the original basis

$$\langle j|i'\rangle = \langle j|\Omega|i\rangle \equiv \Omega_{ji} \tag{1.6.1}$$

are known. The n^2 numbers, Ω_{ij}, are the *matrix elements* of Ω in this basis. If

$$\Omega|V\rangle = |V'\rangle$$

then the components of the transformed ket $|V'\rangle$ are expressable in terms of the Ω_{ij} and the components of $|V\rangle$:

$$\begin{aligned} v_i' = \langle i|V'\rangle = \langle i|\Omega|V\rangle &= \langle i|\Omega\left(\sum_j v_j|j\rangle\right) \\ &= \sum_j v_j \langle i|\Omega|j\rangle \\ &= \sum_j \Omega_{ij} v_j \end{aligned} \tag{1.6.2}$$

Equation (1.6.2) can be cast in matrix form:

$$\begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} = \begin{bmatrix} \langle 1|\Omega|1\rangle & \langle 1|\Omega|2\rangle & \cdots & \langle 1|\Omega|n\rangle \\ \langle 2|\Omega|1\rangle & & & \\ \vdots & & & \vdots \\ \langle n|\Omega|1\rangle & \cdots & & \langle n|\Omega|n\rangle \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{1.6.3}$$

A mnemonic: the elements of the first column are simply the components of the first transformed basis vector $|1'\rangle = \Omega|1\rangle$ in the given basis. *Likewise, the elements of the jth column represent the image of the jth basis vector after Ω acts on it.*

Convince yourself that the same matrix Ω_{ij} acting to the *left* on the row vector corresponding to any $\langle v'|$ gives the row vector corresponding to $\langle v''| = \langle v'|\Omega$.

Example 1.6.1. Combining our mnemonic with the fact that the operator $R(\frac{1}{2}\pi\mathbf{i})$ has the following effect on the basis vectors:

$$R(\tfrac{1}{2}\pi\mathbf{i})|1\rangle = |1\rangle$$

$$R(\tfrac{1}{2}\pi\mathbf{i})|2\rangle = |3\rangle$$

$$R(\tfrac{1}{2}\pi\mathbf{i})|3\rangle = -|2\rangle$$

we can write down the matrix that represents it in the $|1\rangle$, $|2\rangle$, $|3\rangle$ basis:

$$R(\tfrac{1}{2}\pi\mathbf{i}) \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{1.6.4}$$

For instance, the -1 in the third column tells us that R rotates $|3\rangle$ into $-|2\rangle$. One may also ignore the mnemonic altogether and simply use the definition $R_{ij} = \langle i|R|j\rangle$ to compute the matrix. □

Exercise 1.6.1. An operator Ω is given by the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

What is its action?

Let us now consider certain specific operators and see how they appear in matrix form.

(1) The Identity Operator I.

$$I_{ij} = \langle i|I|j\rangle = \langle i|j\rangle = \delta_{ij} \tag{1.6.5}$$

Thus I is represented by a diagonal matrix with 1's along the diagonal. You should verify that our mnemonic gives the same result.

(2) The Projection Operators. Let us first get acquainted with *projection operators*. Consider the expansion of an arbitrary ket $|V\rangle$ in a basis:

$$|V\rangle = \sum_{i=1}^{n} |i\rangle\langle i|V\rangle$$

In terms of the objects $|i\rangle\langle i|$, which are linear operators, and which, by definition, act on $|V\rangle$ to give $|i\rangle\langle i|V\rangle$, we may write the above as

$$|V\rangle=\left(\sum_{i=1}^{n}|i\rangle\langle i|\right)|V\rangle \tag{1.6.6}$$

Since Eq. (1.6.6) is true for all $|V\rangle$, the object in the brackets must be identified with the identity (operator)

$$I=\sum_{i=1}^{n}|i\rangle\langle i|=\sum_{i=1}^{n}\mathbb{P}_i \tag{1.6.7}$$

The object $\mathbb{P}_i=|i\rangle\langle i|$ is called the *projection operator* for the ket $|i\rangle$. Equation (1.6.7), which is called the *completeness relation*, expresses the identity as a sum over projection operators and will be invaluable to us. (If you think that any time spent on the identity, which seems to do nothing, is a waste of time, just wait and see.)

Consider

$$\mathbb{P}_i|V\rangle=|i\rangle\langle i|V\rangle=|i\rangle v_i \tag{1.6.8}$$

Clearly $\mathbb{P}_i$ is linear. Notice that whatever $|V\rangle$ is, $\mathbb{P}_i|V\rangle$ is a multiple of $|i\rangle$ with a coefficient (v_i) which is the component of $|V\rangle$ along $|i\rangle$. Since $\mathbb{P}_i$ projects out the component of any ket $|V\rangle$ along the direction $|i\rangle$, it is called a *projection operator*. The completeness relation, Eq. (1.6.7), says that the sum of the projections of a vector along all the n directions equals the vector itself. Projection operators can also act on bras in the same way:

$$\langle V|\mathbb{P}_i=\langle V|i\rangle\langle i|=v_i^*\langle i| \tag{1.6.9}$$

Projection operators corresponding to the basis vectors obey

$$\mathbb{P}_i\mathbb{P}_j=|i\rangle\langle i|j\rangle\langle j|=\delta_{ij}\mathbb{P}_j \tag{1.6.10}$$

This equation tells us that (1) once $\mathbb{P}_i$ projects out the part of $|V\rangle$ along $|i\rangle$, further applications of $\mathbb{P}_i$ make no difference; and (2) the subsequent application of $\mathbb{P}_j (j\neq i)$ will result in zero, since a vector entirely along $|i\rangle$ cannot have a projection along a perpendicular direction $|j\rangle$.

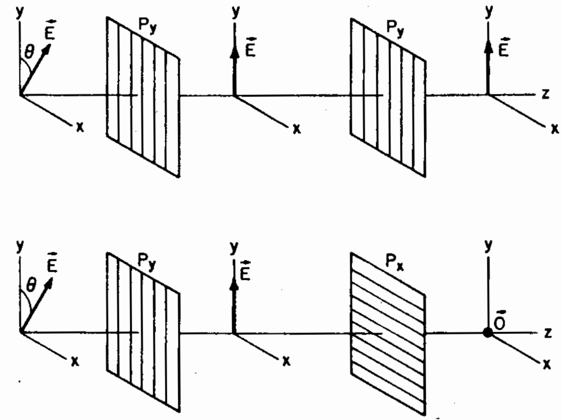

Figure 1.4. P_x and P_y are polarizers placed in the way of a beam traveling along the z axis. The action of the polarizers on the electric field $\mathbf{E}$ obeys the law of combination of projection operators: $P_i P_j = \delta_{ij} P_j$.

The following example from optics may throw some light on the discussion. Consider a beam of light traveling along the z axis and polarized in the $x-y$ plane at an angle θ with respect to the y axis (see Fig. 1.4). If a polarizer P_y, that only admits light polarized along the y axis, is placed in the way, the projection $E\cos\theta$ along the y axis is transmitted. An additional polarizer P_y placed in the way has no further effect on the beam. We may equate the action of the polarizer to that of a projection operator $\mathbb{P}_y$ that acts on the electric field vector $\mathbf{E}$. If P_y is followed by a polarizer P_x the beam is completely blocked. Thus the polarizers obey the equation $P_i P_j = \delta_{ij} P_j$ expected of projection operators.

Let us next turn to the matrix elements of $\mathbb{P}_i$. There are two approaches. The first one, somewhat indirect, gives us a feeling for what kind of an object $|i\rangle\langle i|$ is. We know

$$|i\rangle \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$\langle i| \leftrightarrow [0, 0, \ldots, 1, 0, 0, \ldots, 0]$$

so that

$$|i\rangle\langle i| \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [0, 0, \ldots, 1, 0, \ldots, 0] = \begin{bmatrix} 0 & & & \cdots & & 0 \\ & \ddots & & & & \\ & & 0 & & & \\ \vdots & & & 1 & & \\ & & & & 0 & \\ & & & & & \ddots \\ 0 & & & & & 0 \end{bmatrix} \qquad (1.6.11)$$

by the rules of matrix multiplication. Whereas $\langle V|V'\rangle = (1\times n \text{ matrix}) \times (n\times 1 \text{ matrix}) = (1\times 1 \text{ matrix})$ is a scalar, $|V\rangle\langle V'| = (n\times 1 \text{ matrix}) \times (1\times n \text{ matrix}) = (n\times n \text{ matrix})$ is an operator. The inner product $\langle V|V'\rangle$ represents a bra and ket which have found each other, while $|V\rangle\langle V'|$, sometimes called the *outer product*, has the two factors looking the other way for a bra or a ket to dot with.

The more direct approach to the matrix elements gives

$$(\mathbb{P}_i)_{kl} = \langle k|i\rangle\langle i|l\rangle = \delta_{ki}\delta_{il} = \delta_{kl}\delta_{li} \qquad (1.6.12)$$

which is of course identical to Eq. (1.6.11). The same result also follows from mnemonic. Each projection operator has only one nonvanishing matrix element, a 1 at the ith element on the diagonal. The completeness relation, Eq. (1.6.7), says that when all the $\mathbb{P}_i$ are added, the diagonal fills out to give the identity. If we form the sum over just some of the projection operators, we get the operator which projects a given vector into the subspace spanned by just the corresponding basis vectors.

Matrices Corresponding to Products of Operators

Consider next the matrices representing a product of operators. These are related to the matrices representing the individual operators by the application of Eq. (1.6.7):

$$\begin{aligned}(\Omega\Lambda)_{ij} &= \langle i|\Omega\Lambda|j\rangle = \langle i|\Omega I\Lambda|j\rangle \\ &= \sum_k \langle i|\Omega|k\rangle\langle k|\Lambda|j\rangle = \sum_k \Omega_{ik}\Lambda_{kj}\end{aligned} \qquad (1.6.13)$$

Thus the matrix representing the product of operators is the product of the matrices representing the factors.

The Adjoint of an Operator

Recall that given a ket $\alpha|V\rangle \equiv |\alpha V\rangle$ the corresponding bra is

$$\langle \alpha V| = \langle V|\alpha^* \quad (\text{and } \textit{not } \langle V|\alpha)$$