not parallel by assumption. The intersection point P will determine how much of $|1\rangle$ and $|2\rangle$ we want: we go from the tail of $|3\rangle$ to P using the appropriate multiple of $|1\rangle$ and go from P to the tip of $|3\rangle$ using the appropriate multiple of $|2\rangle$.

Exercise 1.1.4. Consider three elements from the vector space of real 2×2 matrices:

$$|1\rangle=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \qquad |2\rangle=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \qquad |3\rangle=\begin{bmatrix}-2 & -1\\ 0 & -2\end{bmatrix}$$

Are they linearly independent? Support your answer with details. (Notice we are calling these matrices vectors and using kets to represent them to emphasize their role as elements of a vector space.)

Exercise 1.1.5. Show that the following row vectors are linearly dependent: (1, 1, 0), (1, 0, 1), and (3, 2, 1). Show the opposite for (1, 1, 0), (1, 0, 1), and (0, 1, 1).

Definition 4. A vector space has *dimension n* if it can accommodate a maximum of n linearly independent vectors. It will be denoted by $\mathbb{V}^n(R)$ if the field is real and by $\mathbb{V}^n(C)$ if the field is complex.

In view of the earlier discussions, the plane is two-dimensional and the set of all arrows not limited to the plane define a three-dimensional vector space. How about 2×2 matrices? They form a four-dimensional vector space. Here is a proof. The following vectors are linearly independent:

$$|1\rangle=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \qquad |2\rangle=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \qquad |3\rangle=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \qquad |4\rangle=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

since it is impossible to form linear combinations of any three of them to give the fourth any three of them will have a zero in the one place where the fourth does not. So the space is at least four-dimensional. Could it be bigger? No, since any arbitrary 2×2 matrix can be written in terms of them:

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}=a|1\rangle+b|2\rangle+c|3\rangle+d|4\rangle$$

If the scalars a, b, c, d are real, we have a *real four-dimensional space*, if they are complex we have a *complex four-dimensional space*.

Theorem 1. Any vector $|V\rangle$ in an n-dimensional space can be written as a linear combination of n linearly independent vectors $|1\rangle \ldots |n\rangle$.

The proof is as follows: if there were a vector $|V\rangle$ for which this were not possible, it would join the given set of vectors and form a set of $n+1$ linearly independent vectors, which is not possible in an n-dimensional space by definition.

Definition 5. A set of n linearly independent vectors in an n-dimensional space is called a *basis*.

Thus we can write, on the strength of the above

$$|V\rangle = \sum_{i=1}^{n} v_i |i\rangle \tag{1.1.3}$$

where the vectors $|i\rangle$ form a basis.

Definition 6. The coefficients of expansion v_i of a vector in terms of a linearly independent basis ($|i\rangle$) are called the *components of the vector in that basis.*

Theorem 2. The expansion in Eq. (1.1.3) is unique.

Suppose the expansion is not unique. We must then have a second expansion:

$$|V\rangle = \sum_{i=1}^{n} v_i' |i\rangle \tag{1.1.4}$$

Subtracting Eq. (1.1.4) from Eq. (1.1.3) (i.e., multiplying the second by the scalar -1 and adding the two equations) we get

$$|0\rangle = \sum_i (v_i - v_i')|i\rangle \tag{1.1.5}$$

which implies that

$$v_i = v_i' \tag{1.1.6}$$

since the basis vectors are linearly independent and only a trivial linear reiation between them can exist. Note that given a basis the components are unique, but if we change the basis, the components will change. We refer to $|V\rangle$ as the vector in the abstract, having an existence of its own and satisfying various relations involving other vectors. When we choose a basis the vectors assume concrete forms in terms of their components and the relation between vectors is satisfied by the components. Imagine for example three arrows in the plane, $\vec{A}$, $\vec{B}$, $\vec{C}$ satisfying $\vec{A}+\vec{B}=\vec{C}$ according to the laws for adding arrows. So far no basis has been chosen and we do not need a basis to make the statement that the vectors from a closed triangle. Now we choose a basis and write each vector in terms of the components. The components will satisfy $C_i = A_i + B_i$, $i = 1, 2$. If we choose a different basis, the components will change in numerical value, but the relation between them expressing the equality of $\vec{C}$ to the sum of the other two will still hold between the new set of components.

In the case of nonarrow vectors, adding them in terms of components proceeds as in the elementary case thanks to the axioms. If

$$|V\rangle=\sum_i v_i|i\rangle \quad \text{and} \tag{1.1.7}$$

$$|W\rangle=\sum_i w_i|i\rangle \quad \text{then} \tag{1.1.8}$$

$$|V\rangle+|W\rangle=\sum_i (v_i+w_i)|i\rangle \tag{1.1.9}$$

where we have used the axioms to carry out the regrouping of terms. Here is the conclusion:

To add two vectors, add their components.

There is no reference to taking the tail of one and putting it on the tip of the other, etc., since in general the vectors have no head or tail. Of course, if we are dealing with arrows, we can add them either using the tail and tip routine or by simply adding their components in a basis.

In the same way, we have:

$$a|V\rangle=a\sum_i v_i|i\rangle=\sum_i av_i|i\rangle \tag{1.1.10}$$

In other words,

To multiply a vector by a scalar, multiply all its components by the scalar.

1.2. Inner Product Spaces

The matrix and function examples must have convinced you that we can have a vector space with no preassigned definition of length or direction for the elements. However, we can make up quantities that have the same properties that the lengths and angles do in the case of arrows. The first step is to define a sensible analog of the dot product, for in the case of arrows, from the dot product

$$\vec{A}\cdot\vec{B}=|A||B|\cos\theta \tag{1.2.1}$$

we can read off the length of say $\vec{A}$ as $\sqrt{|A|\cdot|A|}$ and the cosine of the angle between two vectors as $\vec{A}\cdot\vec{B}/|A||B|$. Now you might rightfully object: how can you use the dot product to define the length and angles, if the dot product itself requires knowledge of the lengths and angles? The answer is this. Recall that the dot product has a second