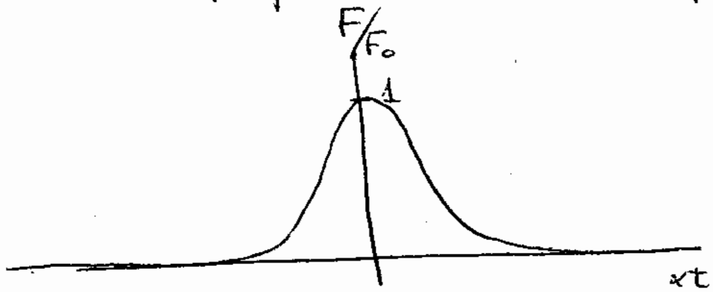
AD: HW#1

1. Given

$$F(t) = \frac{F_0}{1+(\alpha t)^2}$$

a) F_0 has units of force; and α, units of s^{-1}.

b)



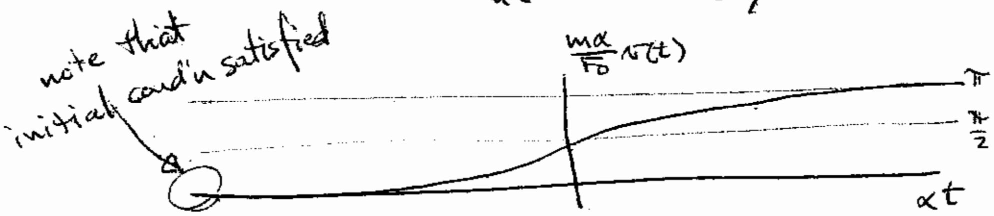
This can represent any kind of impulsive force. That is, any force that is on for a short time only. Any impact, for instance, is usually impulsive.

c) We must do the integral $\left(F = m\dot{v} \Rightarrow \frac{dv}{dt} = \frac{F}{m} \right)$

$$v(t) = \cancel{v(-\infty)}^{\,0} + \int_{-\infty}^{t} dt' \frac{F_0/m}{1+(\alpha t')^2}$$

$$= \frac{F_0}{m\alpha} \int_{-\infty}^{\alpha t} \frac{du}{1+u^2} = \frac{F_0}{m\alpha} \tan^{-1} u \Big|_{-\infty}^{\alpha t}$$

$$= \frac{F_0}{m\alpha}\left(\tan^{-1}\alpha t + \frac{\pi}{2} \right)$$



d) Next,

$$x(t) = \cancel{x(-\infty)}^{\,0} + \int_{-\infty}^{t} dt' v(t')$$

NOTE: $\int dx \tan^{-1} x = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2)$

(Mathematica)

So,

$$x(t) = \int_{-\infty}^{t} dt' \, \frac{F_0}{m\alpha}\left(\tan^{-1}\alpha t + \frac{\pi}{2}\right)$$

$$= \frac{F_0}{m\alpha^2}\int_{-\infty}^{\alpha t} du \left(\tan^{-1} u + \frac{\pi}{2}\right)$$

$$= \frac{F_0}{m\alpha^2}\left[u\tan^{-1}u - \frac{1}{2}\ln(1+u^2) + \frac{\pi}{2}u\right]_{-\infty}^{\alpha t}$$

must be careful about this limit

Actually, the integral doesn't converge for $u \to -\infty$. The problem is that the integrand, $\tan^{-1}u$, behaves like

$$\tan^{-1}u \xrightarrow[u\to-\infty]{} -\frac{\pi}{2} + \frac{1}{u} + \dots$$

So, the integral is essentially

$$x(t) = \int_{-\infty}^{\alpha t} du \, \frac{1}{u} = \ln u \Big|_{-\infty}^{\alpha t}$$

which certainly does not converge at the lower limit. There are two possible ways out of this problem:

i) Take the initial time to be large and negative, but <u>not</u> infinite.

ii) The given force $F(t)$ is not really physical and must go to zero at some point in time faster than the given Lorentzian.

The simpler solution is probably (i).

So, let's take $t=t_0$ to be the initial time:

$$x(t)=\frac{F_0}{m\alpha^2}\Big[\alpha t\tan^{-1}\alpha t-\frac{1}{2}\ln\left(1+(\alpha t)^2\right)+\frac{\pi}{2}\alpha t$$
$$-\alpha t_0\tan^{-1}\alpha t_0+\frac{1}{2}\ln\left[1+(\alpha t_0)^2\right]-\frac{\pi}{2}\alpha t_0\Big]$$

We can approximate

$$\tan^{-1}\alpha t_0\approx-\frac{\pi}{2}-\frac{1}{\alpha t_0}+\ldots$$

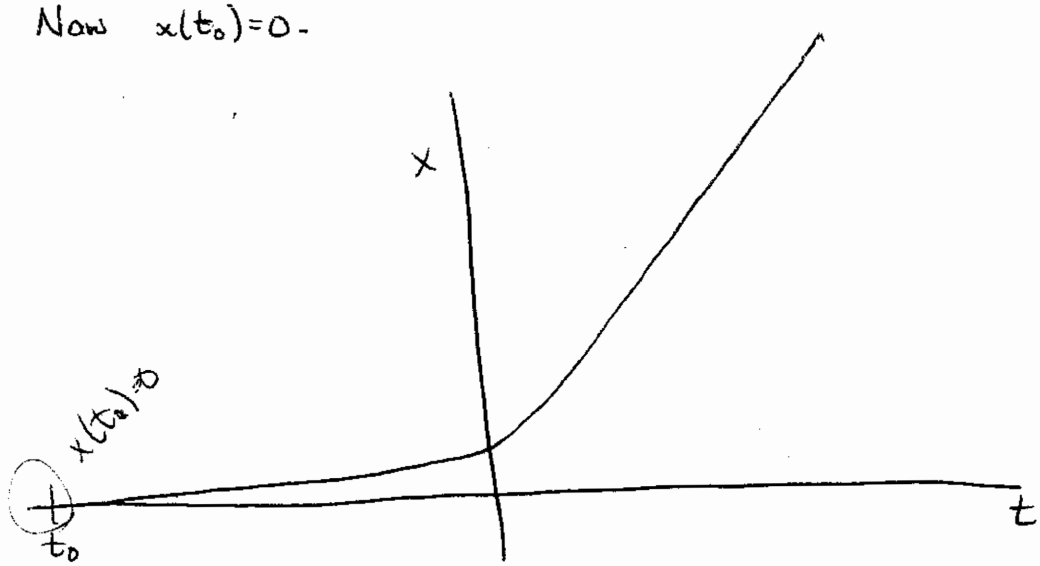
so that

$$x(t)=\frac{F_0}{m\alpha^2}\left[\alpha t\tan^{-1}\alpha t-\frac{1}{2}\ln\left|\frac{1+(\alpha t)^2}{1+(\alpha t_0)^2}\right|+\frac{\pi}{2}\alpha t-\alpha t_0\left(\cancel{-\frac{\pi}{2}}-\frac{1}{\alpha t_0}\right)-\cancel{\frac{\pi}{2}}\alpha t_0\right]$$

This doesn't actually work so well since $t=t_0$ does not give $x=0$ as required. We go back to the equation at the top of the page:

$$x(t)=\frac{F_0}{m\alpha^2}\left[\alpha t\tan^{-1}\alpha t-\alpha t_0\tan^{-1}\alpha t_0-\frac{1}{2}\ln\frac{1+(\alpha t)^2}{1+(\alpha t_0)^2}+\frac{\pi}{2}\alpha(t-t_0)\right]$$

Now $x(t_0)=0$.



The velocity is nearly constant (zero) for early times (long before force is large). The position is also changing slowly.

After the impulse from the force, the velocity again approaches a constant. In fact

$$\Delta v = v(\infty) - v(-\infty)$$
$$= \frac{\pi F_0}{m\alpha}$$

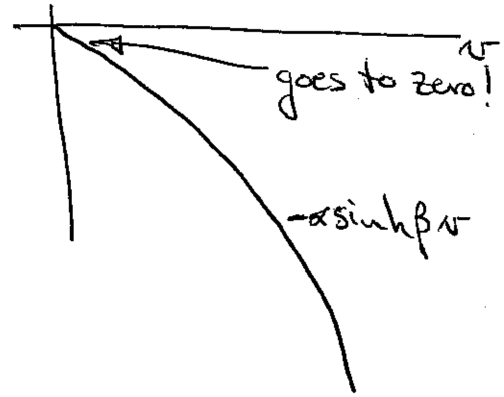
So that

$$\Delta p = \frac{\pi F_0}{\alpha}.$$

The position is growing linearly with time after the force ~~is~~ is past with a slope Δv. Both observations are consistent with an impulse.

2. Given

$$F = -\alpha \sinh \beta v$$



a) The EOM is

$$\dot{v} = -\frac{\alpha}{m} \sinh \beta v$$

$$\int \frac{dv}{\sinh \beta v} = \int -\frac{\alpha}{m}\, dt + C.$$

The tough integral is

$$\int \frac{dx}{\sinh ax} = \frac{1}{a} \ln\left[\tanh\left(\frac{ax}{2}\right)\right] \quad \text{from Mathematica.}$$

So,

$$\int \frac{dv}{\sinh \beta v} = \frac{1}{\beta} \ln\left[\tanh\left(\frac{\beta v}{2}\right)\right] = -\frac{\alpha}{m} t + C$$

$$\Rightarrow \quad \tanh \frac{\beta v}{2} = e^{-\frac{\alpha\beta}{m} t} C'$$

$$v(t) = \frac{2}{\beta} \tanh^{-1}\left[C' e^{-\frac{\alpha\beta}{m} t}\right]$$

The initial condition is

$$v(0) = v_0 = \frac{2}{\beta} \tanh^{-1}[C'] \Rightarrow C' = \tanh\left(\frac{\beta v_0}{2}\right)$$

so that

$$v(t) = \frac{2}{\beta} \tanh^{-1}\left[\tanh\left(\frac{\beta v_0}{2}\right) e^{-\frac{\alpha\beta}{m} t}\right].$$

To get $x(t)$,

choose to be zero

$$x(t) = \cancel{x_0} + \int_0^t dt'\, v(t')$$

$$= \int_0^t dt' \, \frac{2}{\beta} \tanh^{-1}\left[\tanh\left(\frac{\beta v_0}{2}\right) e^{-\frac{\alpha\beta}{m} t}\right].$$

So, we need an integral of the form

$$\int dx \tanh^{-1}\left[a e^{-bx}\right] =$$

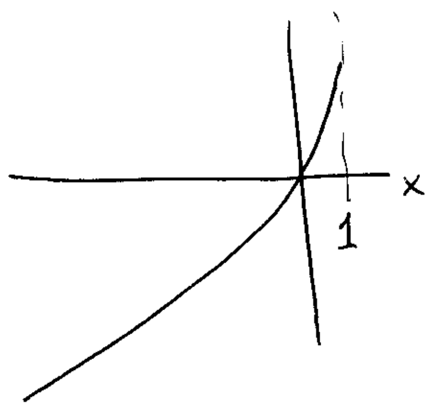
<u>Mathematica</u>!

$$\frac{2bx\left[\tanh^{-1}(ae^{-bx}) - \coth^{-1}(ae^{-bx})\right] - \mathrm{Li}_2\left(-\frac{e^{bx}}{a}\right) + \mathrm{Li}_2\left(\frac{e^{bx}}{a}\right)}{2b}.$$

The functions $\mathrm{Li}_2(x)$ are polylogarithms:

$$\mathrm{Li}_2(x) \equiv -\int_0^x \frac{\log(1-t)}{t}\, dt.$$

It looks like



(see http://dlmf.nist.gov)

It has a branch cut along $x \in [1, \infty)$.

We also have that (defining τ)

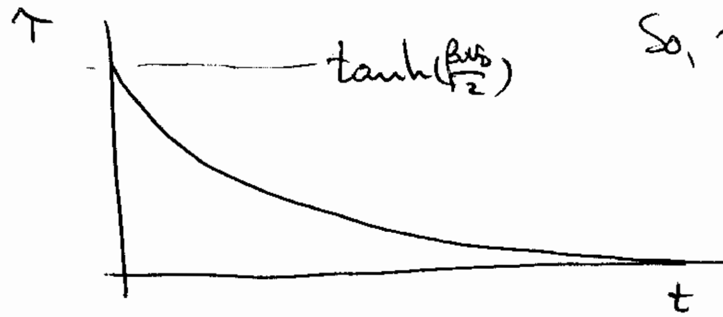
$$\tau \equiv a e^{-bx} = \tanh\left(\frac{\beta v_0}{2}\right) e^{-\frac{\alpha\beta}{m} t} \leq 1, \quad t \geq 0.$$

We thus have

$$x(t) = \frac{2}{\beta}\left[t\left[\tanh^{-1}\tau - \coth^{-1}\tau\right] + \frac{Li_2\left(\frac{1}{\tau}\right) - Li_2\left(-\frac{1}{\tau}\right)}{\frac{2\alpha\beta}{m}} - \frac{Li_2\left(\frac{1}{\tanh\left(\frac{\beta v_0}{2}\right)}\right) - Li_2\left(-\frac{1}{\tanh\left(\frac{\beta v_0}{2}\right)}\right)}{\frac{2\alpha\beta}{m}} \right]$$

which gives $x(t=0)=0$. ~~that~~ Note that the argument of Li_2 here will be larger than 1 (since $\tau \le 1$) so evaluating this takes some care to obtain a real value... see below.

b) Plotting $x(t)$ & $v(t)$ are made a bit more convenient by using the intermediate variable τ. So, let's first note that

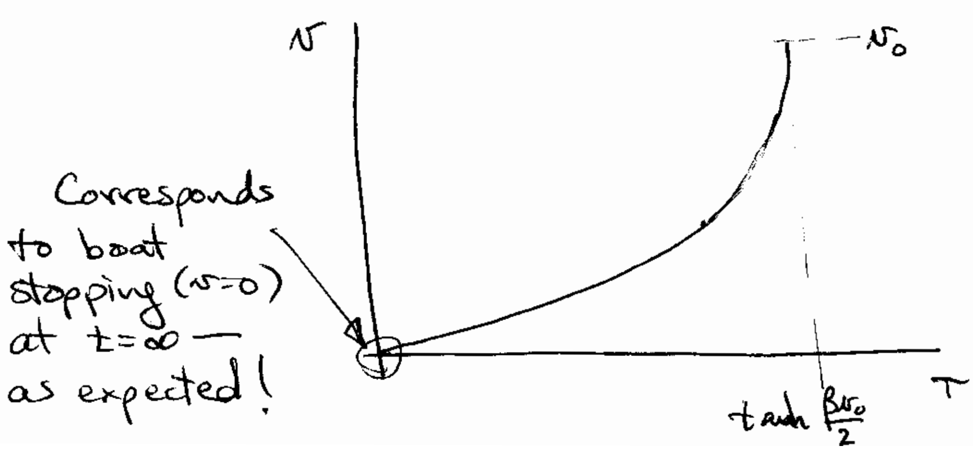


So, $\tau \xrightarrow[t\to\infty]{} 0$

$\tau \xrightarrow[t\to 0]{} \tanh\frac{\beta v_0}{2}$

Thus,

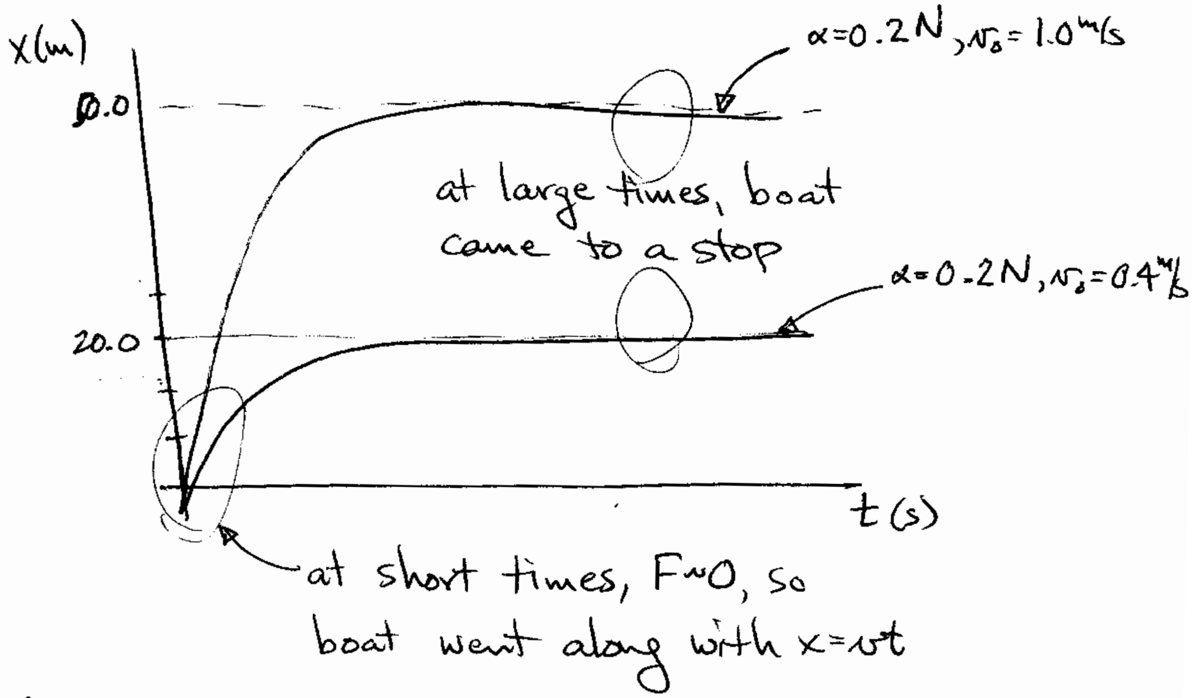
$$v(t) = \frac{2}{\beta}\tanh^{-1}\tau$$



Plotting x(t) still isn't so easy using the analytic expression from p. 7. We know that x(t) must be a real-valued function, but the expression obtained makes this not so easy to get. Since we can't ignore the imaginary part, we have to make sure it's zero. I spent far too long trying to do this and lost the physics in the process. ~~It's~~ It's much easier to simply integrate numerically (Euler's method, for instance).

which is what I used

So, choosing $m = 1.0$ kg, $\beta = 0.1\,(m/s)^{-1}$, I found



Note that "long times" here mean

$$\frac{\alpha\beta}{m} t \gg 1 \quad (\text{arg. of exp. in } v(t)!)$$

$$t \gg \frac{m}{\alpha\beta}$$

Moreover, plotting $v(t)$ for these parameters on a log-linear plot show it to be nearly perfectly exponential. That's because $\tanh^{-1}(*)$ for small $*$ is nearly linear. In the present case,

$$\Upsilon \le \tanh\frac{\beta v_0}{2} \approx \frac{\beta v_0}{2} \quad \text{for } \frac{\beta v_0}{2} \lesssim \frac{1}{2}$$

and thus

$$v(t) \approx \frac{2}{\beta}\tanh\left(\frac{\beta v_0}{2}\right) e^{-\frac{\alpha\beta}{m}t}, \quad \frac{\beta v_0}{2} \lesssim \frac{1}{2}.$$

In this case

$$x(t) = \int_0^t dt' \, \frac{2}{\beta}\tanh\left(\frac{\beta v_0}{2}\right) e^{-\frac{\alpha\beta}{m}t'} + \cancel{x(0)}^{\,0}$$

$$= \frac{2}{\beta}\tanh\left(\frac{\beta v_0}{2}\right)\left(-\frac{m}{\alpha\beta}\right)\left(e^{-\frac{\alpha\beta}{m}t} - 1\right).$$

We thus find that the distance traveled before the boat comes to a stop is

$$x(t \to \infty) = \frac{2m}{\alpha\beta^2}\tanh\left(\frac{\beta v_0}{2}\right)$$

we already assumed this!

$$= \frac{\cancel{2}m}{\alpha\beta^{\cancel{2}}}\frac{\cancel{\beta} v_0}{\cancel{2}}$$

$$= \frac{m v_0}{\alpha\beta} \cdot \left(\frac{\cancel{kg}\cdot m/s^2}{\cancel{kg\, m/s^2}\,(m/s)^{-1}}\right) \checkmark$$

Of course, it takes infinite time to do this!

Note also that this result agrees with the numerically calculated limits on p. 8. The approximation gives a distance of 500 m when $v_0 = 10\,\text{m/s}$, but the actual value is 474 m — so the approximation is starting to break down (~5%) for $\frac{\beta v_0}{2} = \frac{1}{2}$.

c) With thrust the EOM is

$$F = m\dot{v} = F_T - \alpha \sinh \beta v$$

Now, we need (Mathematica)

$$\int \frac{dx}{a \pm b \sinh \alpha x} = -\frac{2}{\alpha i \sqrt{a^2+b^2}} \tan^{-1}\left[\frac{b + a \tanh\frac{\alpha x}{2}}{i\sqrt{a^2+b^2}}\right]$$

$$= \frac{2}{\alpha\sqrt{a^2+b^2}} \tanh^{-1}\left[\frac{b + a\tanh\frac{\alpha x}{2}}{\sqrt{a^2+b^2}}\right].$$

The velocity is thus found from

$$\int \frac{dv}{F_T - \alpha \sinh \beta v} = \frac{1}{m} t + C$$

$$\frac{2}{\beta\sqrt{F_T^2+\alpha^2}} \tanh^{-1}\left[\frac{\alpha + F_T \tanh\left(\frac{\beta v}{2}\right)}{\sqrt{F_T^2+\alpha^2}}\right] = \frac{t}{m} + C$$

$$\alpha + F_T \tanh\left(\frac{\beta v}{2}\right) = \sqrt{F_T^2+\alpha^2} \tanh\left[\sqrt{F_T^2+\alpha^2}\,\frac{\beta t}{2m} + C'\right]$$

$$v(t) = \frac{2}{\beta} \tanh^{-1}\left[\frac{1}{F_T}\left(\sqrt{F_T^2+\alpha^2} \tanh\left[\sqrt{F_T^2+\alpha^2}\,\frac{\beta t}{2m} + C'\right] - \alpha\right)\right]$$

At $t=0$,

$$v(0) = v_0 = \frac{2}{\beta} \tanh^{-1}\left[\frac{1}{F_T}\left(\sqrt{F_T^2+\alpha^2} \tanh C' - \alpha\right)\right]$$

$$F_T \tanh \frac{\beta v_0}{2} + \alpha = \sqrt{F_T^2+\alpha^2} \tanh C'$$

or

$$C' = \tanh^{-1}\left[\frac{F_T \tanh \frac{\beta v_0}{2} + \alpha}{\sqrt{F_T^2+\alpha^2}}\right].$$

Let's see whether we can easily recover the results from (a), i.e. $F_T \to 0$. First,

$$C' = \tanh^{-1} 1 = \infty \quad \ldots \text{ well, not so good.}$$

This doesn't appear to be a nice limit to take. Let's table that idea for now.

To find x(t), we need to integrate—

$$\int du \tanh^{-1}\left[a\left(\tanh(bu+c)+d\right)\right]$$

doesn't look good for an analytic result.
So, let's just write the definition of x(t) and plan to integrate numerically:

$$x(t) = \overset{0}{x_0} + \int^t dt' \frac{2}{\beta}\tanh^{-1}\left[\sqrt{1+\frac{\alpha^2}{F_T^2}}\tanh\left(\sqrt{F_T^2+\alpha^2}\frac{\beta t}{2m}+C'\right)-\frac{\alpha}{F_T}\right].$$

c) We find the terminal velocity from $v(t\to\infty)$:

$$v(t) \xrightarrow[t\to\infty]{} \frac{2}{\beta}\tanh^{-1}\left[\sqrt{1+\frac{\alpha^2}{F_T^2}}-\frac{\alpha}{F_T}\right] = v_t$$

Does this agree with EOM result? Recall at v_t, $\dot{v}=0$, so

$$F_T = \alpha\sinh\beta v_t \Rightarrow v_t = \frac{1}{\beta}\sinh^{-1}\left(\frac{F_T}{\alpha}\right).$$

Hmmm... which is right? The EOM approach is simpler, making it easier to believe.

I can't find any errors in either approach, and they do not seem to be equal, but let's check. Set $x = \frac{\alpha}{F_T}$, so does

$$\frac{2}{\beta}\tanh^{-1}\left[\sqrt{1+x^2}-x\right] = \frac{1}{\beta}\sinh^{-1}\left(\frac{1}{x}\right)?$$

We know each side can be written in terms of logs:

$$\tanh^{-1} y = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) \qquad \sinh^{-1} y = \ln\left(y+\sqrt{1+y^2}\right)$$

So,

$$\cancel{2}\cdot\cancel{\tfrac{1}{2}}\ln\left(\frac{1+\sqrt{1+x^2}-x}{1-\sqrt{1+x^2}+x}\right) \overset{?}{=} \ln\left(\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}\right)$$

$$\frac{1+\sqrt{1+x^2}-x}{1-\sqrt{1+x^2}+x} \overset{?}{=} \frac{1+\sqrt{1+x^2}}{x}$$

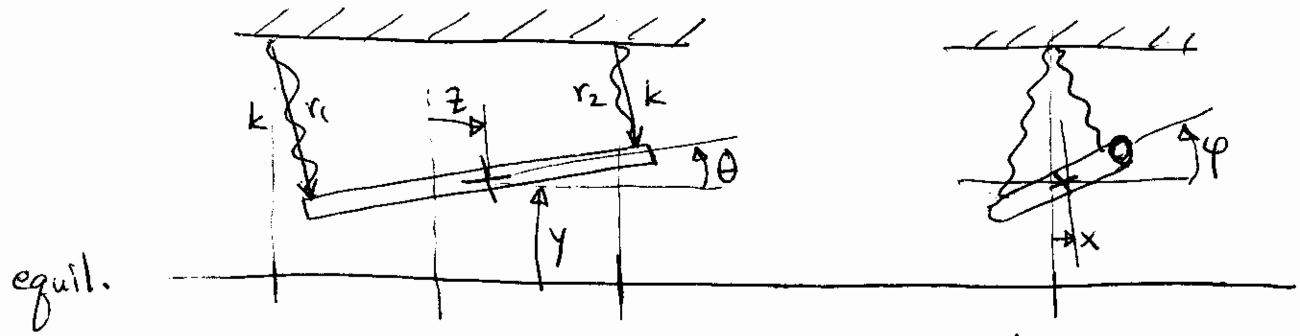
$$x + x\sqrt{1+x^2} - x^2 \overset{?}{=} \left(1+x-\sqrt{1+x^2}\right)\left(1+\sqrt{1+x^2}\right)$$

$$\overset{?}{=} \cancel{1}+x + (\cancel{1}+x)\sqrt{1+x^2} - \cancel{\sqrt{1+x^2}} - \cancel{1} - x^2$$

$$x + x\sqrt{1+x^2} - x^2 = x + x\sqrt{1+x^2} - x^2$$

The two answers are, in fact, the same!

3. We have



— side view —　　　　— end view —

a) The generalized coordinates are x, y, z, θ, φ: x, y, z give the position of the rod's CM with respect to a lab-fixed origin ($x=y=z=0$ puts the CM at its equilibrium position neglecting gravity); θ & φ are the spherical angles of the rod about a lab-fixed coordinate system whose origin is at the rod's CM.

b) So, we have

$$T = \frac{1}{2}M(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 .$$

By symmetry, we know $I_1 = I_2$ (only the moment for the rod's spinning about its axis would be different — and we aren't considering that). We also can write

$$\omega_1 = \dot{\theta} \qquad \omega_2 = \sin\theta\,\dot{\varphi}$$ (actually it's not quite this simple, but the outcome is the same —)

Thus,

$$T = \frac{1}{2}M(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \frac{1}{2}I(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) .$$

Next,

$$U=\frac{1}{2}k\left[(r_1-r_0)^2+(r_2-r_0)^2\right]-Mgy$$

where r_0 is the equilibrium lengths of the springs (without gravity). But,

$$r_1^2=\left(x-\frac{l}{2}\sin\theta\cos\varphi\right)^2+\left(y-\frac{l}{2}\sin\theta\sin\varphi-r_0\right)^2+\left(z-\frac{l}{2}\cos\theta+\frac{l}{2}\right)^2$$

$$r_2^2=\left(x+\frac{l}{2}\sin\theta\cos\varphi\right)^2+\left(y+\frac{l}{2}\sin\theta\sin\varphi-r_0\right)^2+\left(z+\frac{l}{2}\cos\theta-\frac{l}{2}\right)^2.$$

To be consistent with the implicit assumptions we are making about our springs, we make the small oscillation approximation (all displacements $\ll r_0$). For simplicity, we write

$$r_1^2=\rho_1^2+(y_1-r_0)^2$$

$$r_2^2=\rho_2^2+(y_2-r_0)^2,$$

giving

$$(r_1-r_0)^2+(r_2-r_0)^2\approx\left(\sqrt{\rho_1^2+(y_1-r_0)^2}-r_0\right)^2+\left(\sqrt{\rho_2^2+(y_2-r_0)^2}-r_0\right)^2$$

$$\approx y_1^2+y_2^2.$$

Finally,

$$U=\frac{1}{2}k\left(y_1^2+y_2^2\right)-Mgy$$

$$=\frac{1}{2}k\left[\left(y-\frac{l}{2}\sin\theta\sin\varphi\right)^2+\left(y+\frac{l}{2}\sin\theta\sin\varphi\right)^2\right]-Mgy$$

$$=\frac{1}{2}k\left[2y^2+\frac{l^2}{4}\sin^2\theta\sin^2\varphi\right]-Mgy.$$

Now, as usual, we combine the gravitational potential with ky^2, complete the square, and define a new displacement coordinate

$$y' = y - y_{eq}$$ includes gravity.

(we also ignore the constant introduced when completing the square).

Thus,

$$L = T - U$$
$$= \frac{1}{2}M(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \frac{1}{2}I(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2)$$
$$- \frac{1}{2}(2k)\left(y'^2 + \frac{\ell^2}{4}\sin^2\theta\sin^2\varphi\right).$$

The EOM are

$\underline{x}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \longrightarrow M\ddot{x} = 0$

$\underline{y'}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}'}\right) - \frac{\partial L}{\partial y'} = 0 \longrightarrow M\ddot{y}' + 2ky' = 0$

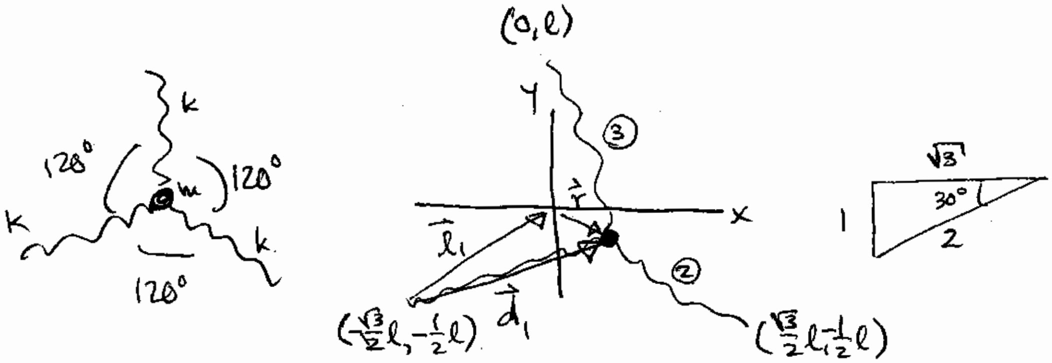
$\underline{z}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \longrightarrow M\ddot{z} = 0$

$\underline{\theta}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \longrightarrow I\ddot{\theta} - 2\cdot\frac{1}{2}I\sin\theta\cos\theta\,\dot{\varphi}^2 + \frac{1}{2}\cdot 2\cdot 2k\frac{\ell^2}{4}\sin\theta\cos\theta\sin^2\varphi = 0$

$\underline{\varphi}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0 \longrightarrow \frac{d}{dt}\left(I\sin^2\theta\,\dot{\varphi}\right) + 2k\frac{\ell^2}{4}\sin^2\theta\sin\varphi\cos\varphi = 0$

c) We have two cyclic coordinates: x & z. It's clear, though, that the conservation of their conjugate momenta is an artifact of the small oscillation approximation. That is, the full potential definitely depends on both x & z.

6.



a) (x, y) — displacement from equilibrium

b) $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$$U = \frac{1}{2}k\left[(d_1 - l_1)^2 + (d_2 - l_2)^2 + (d_3 - l_3)^2\right]$$

$l_1 = l_2 = l_3 = l$ is relaxed length of spring.

So, we need

$$d_1 = \left[\left(x + \frac{\sqrt{3}}{2}l\right)^2 + \left(y + \frac{1}{2}l\right)^2\right]^{\frac{1}{2}}$$

$$d_2 = \left[\left(x - \frac{\sqrt{3}}{2}l\right)^2 + \left(y + \frac{1}{2}l\right)^2\right]^{\frac{1}{2}}$$

$$d_3 = \left[x^2 + (y - l)^2\right]^{\frac{1}{2}}.$$

The idea is to expand $(d_i - l)^2$ to terms quadratic in x, y.

Using Mathematica:

$$(d_1 - l)^2 \approx \frac{1}{4}y^2 + \frac{\sqrt{3}}{2}xy + \frac{3}{4}x^2$$

$$(d_2 - l)^2 \approx \frac{1}{4}y^2 - \frac{\sqrt{3}}{2}xy + \frac{3}{4}x^2$$

$$(d_3 - l)^2 \approx y^2$$

$$\Rightarrow U = \frac{1}{2}k\left[\frac{3}{2}y^2 + \frac{3}{2}x^2\right]$$

coupling vanishes due to symmetry of problem!

Thus,

$$L = T - U$$
$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k\left(\frac{3}{2}\right)(x^2 + y^2).$$

c) There are no cyclic coordinates.

5. a) $F = qE(t) = -eE(t) = m\frac{dv}{dt}$

So,

$$v = -\frac{e}{m}\int_0^t dt'\, E_0 \tanh\gamma t' + v(0)$$

$$= -\frac{eE_0}{m}\int_0^t dt' \frac{\sinh\gamma t'}{\cosh\gamma t'} + v(0)$$

$u = \cosh\gamma t \Rightarrow du = \gamma \sinh\gamma t\, dt$

$$\Rightarrow v(t) = -\frac{eE_0}{m\gamma}\int_1^{\cosh\gamma t} \frac{du}{u} + v(0)$$

$$= -\frac{eE_0}{m\gamma}\left[\ln\cosh\gamma t - \cancel{\ln 1}^{\,0}\right] + v(0)$$

Finally,

$$v(t) = v_0 - \frac{eE_0}{m\gamma}\ln\cosh\gamma t.$$

Check: $v(t\to 0) = v_0 - \frac{eE_0}{m\gamma}\ln\cosh 0 = v_0$ ✓.

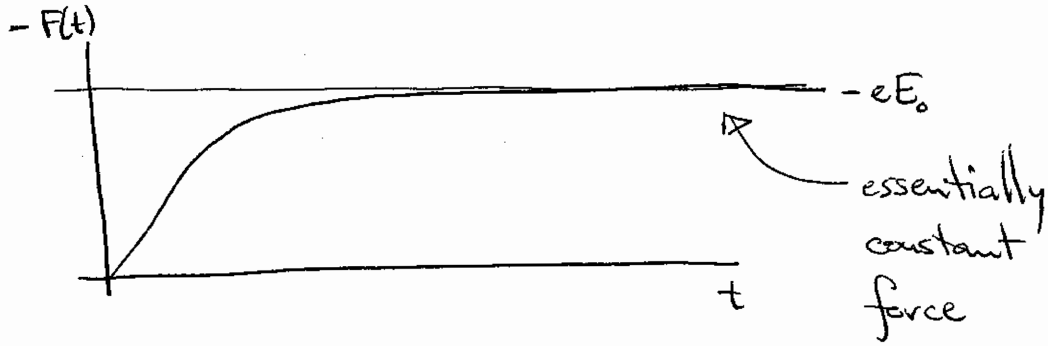
b) For $t\to\infty$, $\cosh\gamma t \to \frac{1}{2}e^{\gamma t}$ and

$$v(t) \xrightarrow[t\to\infty]{} v_0 - \frac{eE_0}{m\gamma}\ln\frac{1}{2}e^{\gamma t}$$

$$= v_0 - \frac{eE_0}{m\gamma}\left(\ln\frac{1}{2} + \gamma t\right)$$

$$= v_0 + \frac{eE_0}{m\gamma}\ln 2 - \frac{eE_0}{m}t.$$

So, for large times, the speed is increasing linearly with time. This should be expected since the force behaves as:



We know that a constant force gives a velocity of the form $v = v_0 + \frac{E}{m}t$, so this behavior does make sense. For t to be considered large (and cosh take on its asymptotic form), we must have

$$\gamma t \gg 1 \Rightarrow t \gg \frac{1}{\gamma}.$$

c) Using Mathematica, we find

$$\int_0^t dx \ln\cosh\gamma x = \frac{1}{2\gamma}\Big[-\gamma t\left(\gamma t + 2\ln(1+e^{-2\gamma t}) - 2\ln(\cosh\gamma t)\right) + \mathrm{PolyLog}[2, -e^{-2\gamma t}]\Big] + \frac{\pi^2}{24\gamma}$$

$$= \frac{1}{2\gamma}\left[\frac{\pi^2}{12} - \gamma^2 t^2 - 2\gamma t \ln(1+e^{-2\gamma t}) + 2\gamma t \ln(\cosh\gamma t) + Li_2(-e^{-2\gamma t})\right]$$

Also find useful,

$$Li_2(0) = 0 \qquad Li_2(-1) = -\frac{\pi^2}{12}$$

$$Li_2(x) \xrightarrow[x \to 0]{} x + \frac{1}{4}x^2$$

So,

$$x(t) = x_0 + \int_0^t dt'\, v(t')$$

$$= x_0 + v_0 t - \frac{eE_0}{m\gamma}\int_0^t dt'\, \ln\cosh\gamma t'$$

$$= x_0 + v_0 t - \frac{eE_0}{m\gamma}\frac{1}{2\gamma}\Big[\frac{\pi^2}{12} - \gamma^2 t^2 - 2\gamma t \ln(1+e^{-2\gamma t})$$
$$+ 2\gamma t \ln(\cosh\gamma t) + \mathrm{Li}_2(-e^{-2\gamma t})\Big].$$

When $t=0$, $x=x_0$ as it should. When $t\to\infty$,

$$x(t) \xrightarrow[t\to\infty]{} x_0 + v_0 t - \frac{eE_0}{2m\gamma^2}\Big[\frac{\pi^2}{12} - \gamma^2 t^2 - 2\gamma t \overset{0}{\cancel{\ln 1}} + 2\gamma t \ln\frac{e^{\gamma t}}{2} + \overset{0}{\cancel{\mathrm{Li}_2(0)}}\Big]$$

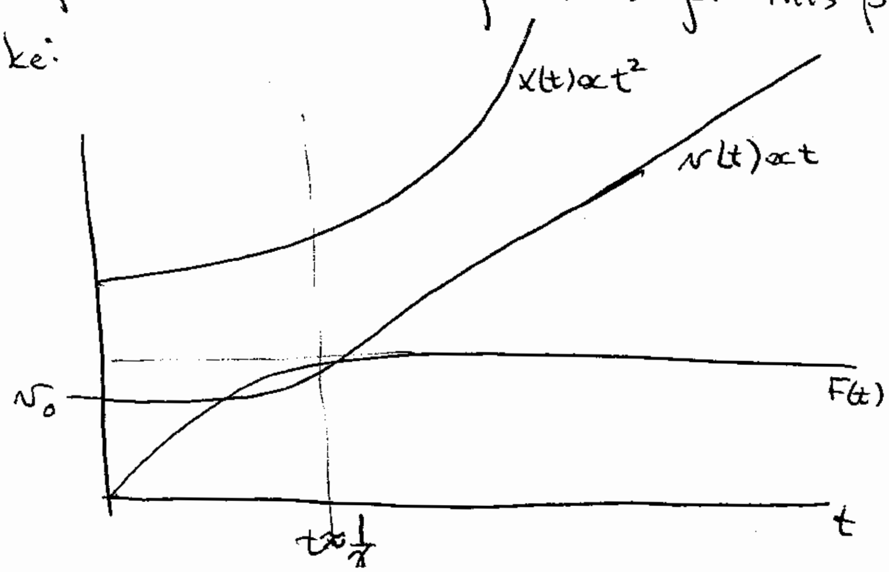
$$= x_0 + v_0 t - \frac{eE_0}{2m\gamma^2}\Big[\frac{\pi^2}{12} - \gamma^2 t^2 - 2\gamma t \ln 2 + 2\gamma^2 t^2\Big]$$

$$= x_0 + v_0 t - \frac{eE_0}{2m}t^2 - \frac{eE_0}{2m\gamma^2}\Big(\frac{\pi^2}{2} - 2\gamma t \ln 2\Big).$$

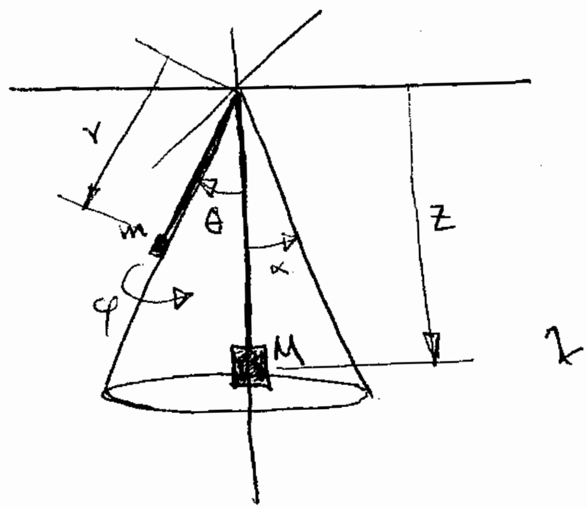
The dominant term is

$$x(t) \xrightarrow[t\to\infty]{} -\frac{1}{2}\Big(\frac{eE_0}{m}\Big)t^2$$

which is just what one would expect for a constant negative force. The various functions for this problem look like:



6. Given



a) There are four degrees of freedom which we will give the GCs (r, θ, φ, z) — see diagram. These are not all independent, though, since

$$r+z=d.$$

We also include θ even though the mass m stays on the cone since later in the problem we want to find out about when it leaves. This coord is constrained by

$$\theta=\alpha.$$

We don't care about the string, though, so the first constraint will be used from the beginning.

The Lagrangian is $L = T - U$,

$$T = \frac{1}{2} M \dot{z}^2 + \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2 \right)$$

$$= \frac{1}{2} M \dot{r}^2 + \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2 \right) \quad \text{(using } r + z = d\text{)}.$$

2

$$U = -mgr\cos\theta - Mgz$$

$$= -mgr\cos\theta - Mg(d - r). \quad 2$$

b) The equations of motion are

$\underline{r}$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mr\sin^2\theta \, \dot{\varphi}^2 + mg\cos\theta - Mg$$

$$\frac{\partial L}{\partial \dot{r}} = (M+m)\dot{r}$$

2

$$\Rightarrow mr\dot{\theta}^2 + mr\sin^2\theta \, \dot{\varphi}^2 + Mg + mg\cos\theta - (M+m)\ddot{r} = 0$$

$\underline{\theta}$

$$\frac{\partial L}{\partial \theta} = m \sin\theta \cos\theta \, r^2 \dot{\varphi}^2 - mgr\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta}$$

2

$$\Rightarrow mr^2 \sin\theta\cos\theta \, \dot{\varphi}^2 - mgr\sin\theta - \frac{d}{dt}\left(mr^2\dot{\theta}\right) = \lambda$$

φ

$$\frac{\partial L}{\partial \varphi} = 0 \qquad \frac{\partial L}{\partial \dot{\varphi}} = mr^2 \sin^2\theta \, \dot{\varphi}$$

2 1

$$\Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = \text{constant} = l_z$$

$\therefore$ φ is cyclic and $p_\varphi = \frac{\partial L}{\partial \dot{\varphi}}$ is the angular momentum about z-axis

c) Need to solve eq'ns of motion to find λ. Using constraint $\theta = \alpha$, $\dot{\theta} = 0$, so θ-EOM is

$$mr^2 \sin\alpha \cos\alpha \, \dot{\varphi}^2 - mgr\sin\alpha = \lambda$$

$$\frac{l_z^2 \cos\alpha}{mr^2 \sin^3\alpha} - mgr\sin\alpha = \lambda \quad 2$$

Now, λ must represent something like a "normal torque". When it is zero, mass m will leave the surface:

$$\frac{l_z^2 \cos\alpha}{mr^2 \sin^3\alpha} = mgr\sin\alpha$$

$$l_z^2 \cos\alpha = m^2 g r^3 \sin^4\alpha$$

or

$$\frac{l_z^2}{r^3} = m^2 g \tan\alpha \sin^3\alpha .$$

This collects dynamical variables on the left, and constants on the right. To understand this, it would help to be able to write an inequality rather than an equality. To do this, we have to use a little physics since the sign of λ is not ~~well~~ uniquely defined (we could get $\pm\lambda$ by writing the constraint as $\theta - \alpha = 0$ or $\alpha - \theta = 0$). So, when do we expect mass m to leave the surface? When $\dot{\varphi}$ (or l_z) is large.

Increasing l_z must correspond to making m leave the surface. So, we should write

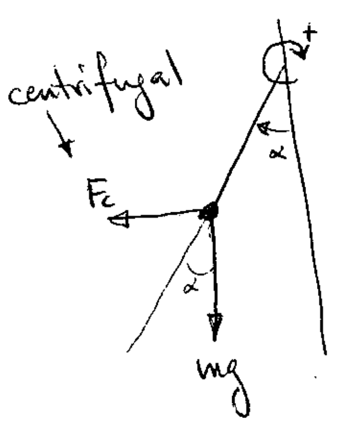
$$\frac{l_z^2\cos\alpha}{mr^2\sin^3\alpha} \geq mgr\sin\alpha$$

as the condition to leave the surface. This means that λ is actually the net torque on mass m. When $\lambda > 0$, m will accelerate in positive θ direction (off of surface!); when $\lambda \leq 0$, m rests on surface. So, finally, we can write

$$\frac{l_z^2}{r^3} \geq m^2 g \tan\alpha \sin^3\alpha \,.$$

If we increase l_z at fixed r, then mass leaves surface. If we decrease r at fixed l_z, m leaves surface. Both of these make sense.

Let's verify by summing torques directly. When torque from F_c overcomes gravity, m leaves surface. So,



$$\tau_c = F_c r\cos\alpha = m\dot{\varphi}^2 r^2 \sin^2\alpha \, r\cos\alpha$$

$$\tau_g = -mgr\sin\alpha$$

+3

$$|\tau_c| \geq |\tau_g| \Rightarrow m\dot{\varphi}^2 r^2 \sin\alpha\cos\alpha \geq mgr\sin\alpha$$

Substituting l_z gives

$$\frac{l_z^2\cos\alpha}{mr^2\sin^3\alpha} \geq mgr\sin\alpha$$

as before. Note also that it's now easy to see that

$$\tau_{net} = \tau_c + \tau_g = m\dot{\varphi}^2 r^2\sin\alpha\cos\alpha - mgr\sin\alpha = \lambda.$$

d) The only coordinate left is r. So, its EOM is $(\theta = \alpha)$

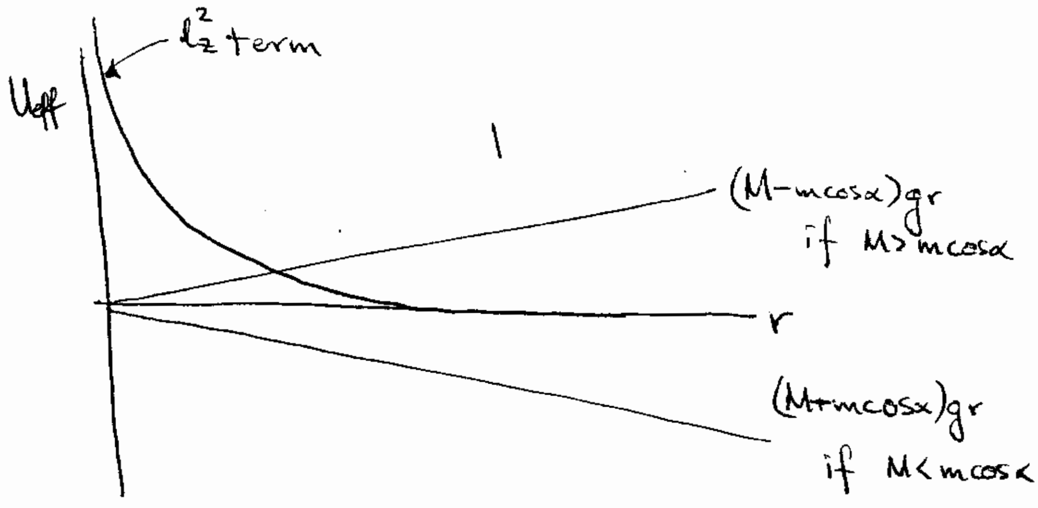
$$mr\sin^2\alpha\,\dot{\varphi}^2 + mg\cos\alpha - Mg = (M+m)\ddot{r}$$

$$\frac{l_z^2}{mr^3\sin^2\alpha} + mg\cos\alpha - Mg = (M+m)\ddot{r}. \quad 1$$

The RHS is the net force, so U_{eff} is defined from

$$U_{eff} = -\int^r\left(\frac{l_z^2}{mr'^3\sin^2\alpha} + mg\cos\alpha - Mg\right)dr'$$

$$= \frac{l_z^2}{2mr^2\sin^2\alpha} + (M - m\cos\alpha)gr \quad 2$$

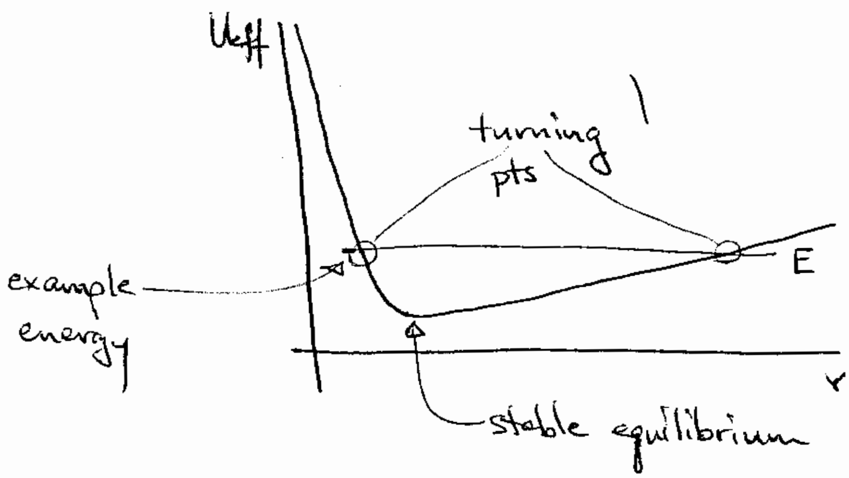


This potential looks like many others we've come across. It shows that if l_z is large, mass m will generally stay far from the apex of the cone ($r=0$) which makes sense. If $M > m\cos\alpha$, then r wants to shrink (since M will pull m towards the apex). Only if $l_z \neq 0$ will this tendency be stopped. Otherwise, mass m will be pulled through the apex of the cone. If $M < m\cos\alpha$, then m tries to pull M through the apex, $l_z \neq 0$ only exacerbates this.

So, the only way for M to be stationary is if

$$M \geq m\cos\alpha.$$

The equality part of this is only possible if $l_z = 0$, meaning no motion at all. If $M > m\cos\alpha$, then the motion is



only bound motion possible (assuming infinite cone) ~~and neglect~~; system will oscillate between two turning points

So, as system oscillates, mass m goes up, M down and vice versa. Simultaneously, m will be going around cone. In principle, can only describe cases here where $r<d$ — otherwise M bangs into apex of cone.

Note that stable equilibrium and small oscillations possible <u>only</u> when m has angular momentum.

e) Assuming $l_z \neq 0$ and $M > m\cos\alpha$, then the stable equilibrium is at

$$(M+m)\ddot{r} = 0 = \frac{l_z^2}{m r_0^3 \sin^2\alpha} + mg\cos\alpha - Mg$$

$$\Rightarrow r_0 = \left[(M - m\cos\alpha) g \frac{m\sin^2\alpha}{l_z^2}\right]^{-\frac{1}{3}} \quad 2$$

This, by the way, shows ~~that~~ mathematically why $M \geq m\cos\alpha$. If it weren't $r_0 < 0$ or imaginary. Note also that $M = m\cos\alpha \Rightarrow r_0 = \infty$ unless $l_z = 0$ also.

For small oscillations, we want displacement. So, define

$$r = r_0 + \Delta r.$$

The EOM is then

$$(M+m)\,\ddot{\Delta r} = \frac{l_z^2}{m(r_0+\Delta r)^3\sin^2\alpha} + mg\cos\alpha - Mg$$

$$= \frac{l_z^2}{m r_0^3 \sin^2\alpha}\left(1+\frac{\Delta r}{r_0}\right)^{-3} + mg\cos\alpha - Mg$$

small oscillations means $\frac{\Delta r}{r_0} \ll 1$, so expand

$$(M+m)\,\ddot{\Delta r} = \frac{l_z^2}{m r_0^3 \sin^2\alpha}\left(\cancel{1} - 3\frac{\Delta r}{r_0} + \ldots\right) + \cancel{mg\cos\alpha} - \cancel{Mg}$$

cancellations from def. of r_0.

$$(M+m)\,\ddot{\Delta r} = -\frac{3 l_z^2}{m r_0^4 \sin^2\alpha}\,\Delta r \qquad \underline{\text{Hooke's Law}}$$

So,

$$\omega = \sqrt{\frac{3 l_z^2}{m r_0^4 \sin^2\alpha\,(M+m)}} \quad . \quad 2$$

This immediately gives

$$r(t) = r_0 + A e^{i\omega t} + B e^{-i\omega t} \qquad (\Delta r = r - r_0).$$

$$\theta(t) = \alpha$$

$$\dot{\varphi}(t) = \frac{l_z}{m r^2 \sin^2\alpha} \implies \cancel{m\sin^2\alpha}\ \varphi(t) = \frac{l_z}{m\sin^2\alpha}\int_0^t \frac{dt'}{r^2(t')} + \varphi(0).$$

$$z(t) = d - r(t).$$

If we write $r(t)$ as

$$r(t)=r_0+A\cos(\omega t-\delta)$$

then

$$\varphi(t)=\varphi(0)+\frac{l_z}{m\sin^2\alpha}\left[\,2r_0\tanh^{-1}\right.$$ .... complicated, but analytical (Mathematica).

We know, though, that $A\ll r_0$ (small oscillations). So, expanding this result for $\frac{A}{r_0}\ll 1$ gives (Mathematica)

$$\varphi(t)=\varphi(0)+\frac{l_z}{m\sin^2\alpha}\left[-\frac{\omega t-\delta}{r_0^2\omega}+\left[\frac{\sin(\omega t-\delta)}{r_0^3\omega}+\frac{2\tan\left(\frac{\omega t-\delta}{2}\right)}{r_0^3\omega\left(1+\tan^2\left(\frac{\omega t-\delta}{2}\right)\right)}\right]A\right]$$

to ~~lowest~~ linear order in A, or

$$\varphi(t)=\varphi(0)-\frac{l_z}{m\omega r_0^2\sin^2\alpha}\left[\omega t-\delta-\left[\sin(\omega t-\delta)+2\sin\left(\frac{\omega t-\delta}{2}\right)\cos\left(\frac{\omega t-\delta}{2}\right)\right]\frac{A}{r_0}\right]$$

$$=\varphi(0)-\frac{l_z}{m\omega r_0^2\sin^2\alpha}\left[\omega t-\delta-2\sin(\omega t-\delta)\frac{A}{r_0}\right].$$

To leading order in $\frac{A}{r_0}$, this is simply

$$\varphi(t)=\varphi(0)-\frac{l_z}{m\omega r_0^2\sin^2\alpha}(\omega t-\delta).$$

We can thus compare the motion in φ to that in r by defining

$$\Omega=\frac{l_z}{mr_0^2\sin^2\alpha}\quad\Rightarrow\quad\varphi(t)=\varphi(0)-\Omega t+\frac{\Omega}{\omega}\delta$$

The ratio of frequencies will tell us the relative rate of oscillations in r & φ:

$$\frac{\omega}{\Omega} = \sqrt{\frac{3\cancel{l_z^2}}{m\cancel{r_0^4}\cancel{\sin^4\alpha}(M+m)}}\;\frac{m\cancel{r_0^2}\cancel{\sin^2\alpha}}{\cancel{l_z^2}}$$

$$= \sqrt{3\frac{m}{M+m}}\,\sin\alpha.$$ Doesn't depend on l_z!

Can we ever make this larger than unity?

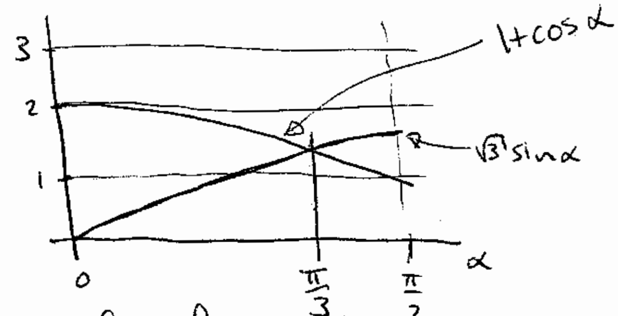
$$\sqrt{3\frac{1}{1+\frac{M}{m}}}\,\sin\alpha > 1$$

$$\sqrt{3}\sin\alpha > 1+\frac{M}{m}$$

The smallest we can make M is $m\cos\alpha$:

$$\sqrt{3}\sin\alpha > 1+\cos\alpha$$

Can we satisfy this inequality?

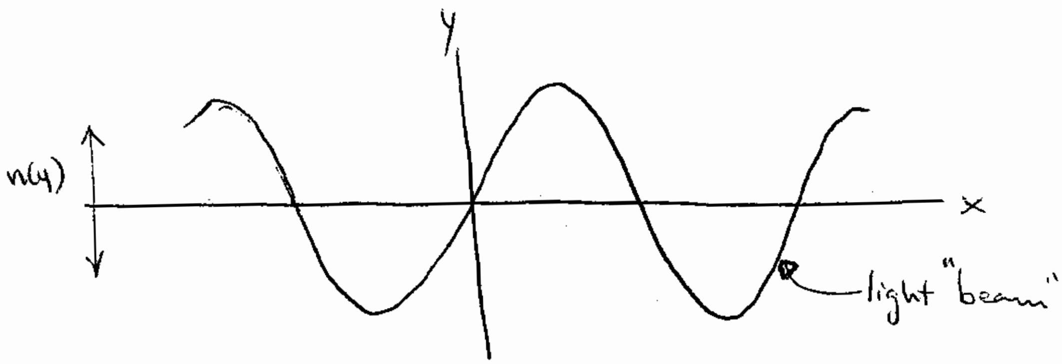


Apparently we can for fairly flat cones. But there will be no oscillations for $M=m\cos\alpha$. So, $\alpha>\frac{\pi}{3}$ will give faster radial oscillations than angular when $M>m\cos\alpha$.

3. a) We're given

$$y(x) = y_0 \sin kx$$

for the path of the light. In other words



We're also told there's an index of refraction gradient in the y-direction. As in the HW and class example, the principle of least time means minimizing

$$t = \int_{x_1}^{x_2} dx \, \frac{\sqrt{1+(y')^2}}{v} \qquad +3$$

but $v = v(y) = \dfrac{c}{n(y)}$. So,

$$t = \int_{x_1}^{x_2} dx \, \frac{\sqrt{1+(y')^2}\, n(y)}{c}. \qquad 2$$

Applying the Euler eq'n gives $\left(f = \sqrt{1+(y')^2}\, n(y)\right)$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \sqrt{1+(y')^2}\,\frac{\partial n}{\partial y} - \frac{d}{dx}\left(\frac{y' n(y)}{\sqrt{1+(y')^2}}\right) = 0 \qquad 1$$

n depends only on y

$$\sqrt{1+(y')^2}\,\frac{dn}{dy} - \frac{\partial n}{\partial y}\frac{dy}{dx}\frac{y'}{\sqrt{1+(y')^2}} - n(y)\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

$$\frac{1}{\sqrt{1+(y')^2}}\frac{dn}{dy} - n\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

But, we're solving for n, not y — we know y:

$$y = y_0 \sin kx \qquad y' = ky_0 \cos kx.$$

So,

$$\sqrt{1+(y')^2} = \sqrt{1+k^2y_0^2\cos^2 kx}$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = \frac{d}{dx}\left(\frac{ky_0\cos kx}{\sqrt{1+k^2y_0^2\cos^2 kx}}\right)$$

$$= \frac{-k^2y_0\sin kx}{\sqrt{1+k^2y_0^2\cos^2 kx}} + \frac{\frac{1}{2}ky_0\cos kx\, k^3y_0^2\cos kx \sin kx}{(1+k^2y_0^2\cos^2 kx)^{\frac{3}{2}}}$$

$$= \frac{-k^2y_0\sin kx}{(1+k^2y_0^2\cos^2 kx)^{\frac{3}{2}}},$$

We need to put this in terms of y, though, so

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = \frac{-k^2y}{\left[1+k^2y_0^2\left(1-\left(\frac{y}{y_0}\right)^2\right)\right]^{\frac{3}{2}}}.$$

Everything together gives

$$\frac{dn}{dy} = -n\frac{k^2y}{1+k^2y_0^2\left(1-\left(\frac{y}{y_0}\right)^2\right)}.$$

This is a separable, first order equation in y:

$$\int \frac{dn}{n} = -\int \frac{k^2 y\, dy}{1+k^2 y_0^2\left[1-\left(\frac{y}{y_0}\right)^2\right]} \qquad \begin{aligned} u &= 1+k^2 y_0^2\left[1-\left(\tfrac{y}{y_0}\right)^2\right] \\ du &= -2k^2 y\, dy \end{aligned}$$

integration constant

$$\ln \frac{n}{n_0} = \frac{1}{2}\int \frac{du}{u}$$

$$\Rightarrow n = n_0\sqrt{u} = n_0\sqrt{1+k^2 y_0^2\left[1-\left(\tfrac{y}{y_0}\right)^2\right]}\,.$$

Certainly this is okay for $y<y_0$. If $y>y_0$, may have some problems: n must be real:

$$1+k^2 y_0^2\left[1-\left(\tfrac{y}{y_0}\right)^2\right] \geq 0$$

$$1+k^2 y_0^2 \geq k^2 y^2$$

$$\Rightarrow |y| \leq \sqrt{\frac{1+k^2 y_0^2}{k^2}}\,.$$

This range includes y_0, so we're okay. Actually, our constraint is a little tighter since we ~~were~~ need $n \geq 1$.

$$n_0\sqrt{1+k^2 y_0^2\left[1-\left(\tfrac{y}{y_0}\right)^2\right]} \geq 1$$

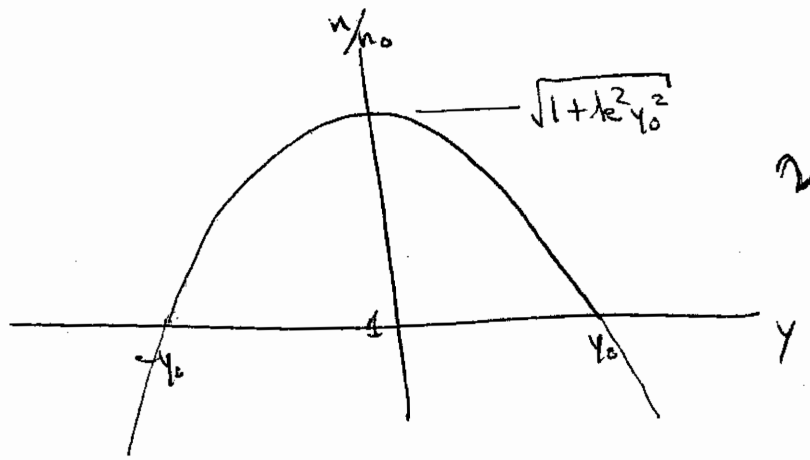
$$1+k^2 y_0^2\left[1-\left(\tfrac{y}{y_0}\right)^2\right] \geq \frac{1}{n_0^2}$$

$$1-\frac{1}{n_0^2}+k^2 y_0^2 \geq k^2 y^2$$

$$|y| \leq \sqrt{\frac{1+k^2 y_0^2-\frac{1}{n_0^2}}{k^2}}\,.$$

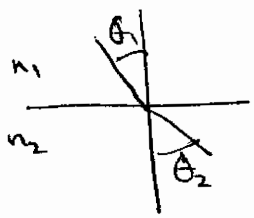
If $n_o = 1$, then $|y| \le y_o$; if $n_o > 1$, then $|y|$ can be larger than y_o, so all is still well.

b)



So, the index is low near the edges — n_o at $\pm y_o$.

If we just look at Snell's law:



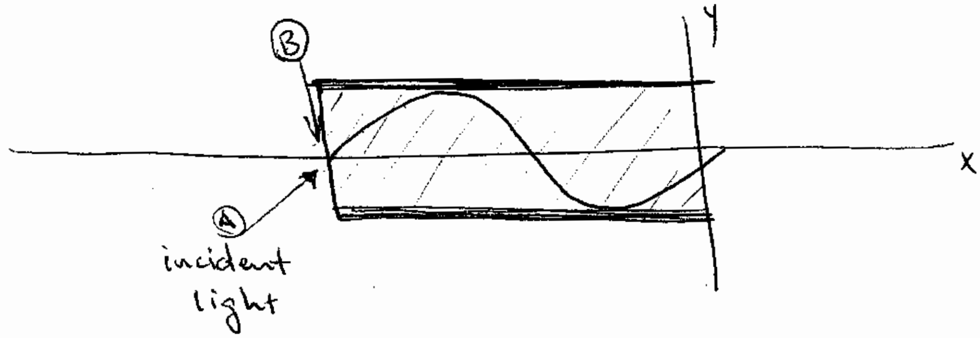
$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$\sin\theta_2 \to 1 \Rightarrow \theta_2 \to \frac{\pi}{2} \Rightarrow$ total internal reflection in region 1

$$\therefore \quad \sin\theta_1 = \frac{n_2}{n_1}$$

For light to be totally internally reflected — which is just a little more extreme than the case we're considering — $\frac{n_2}{n_1} < 1$. In other words, light gets strongly bent going from high index regions to low. So, our $n(y)$ is completely reasonable.

c) If we imagine the device's end:



Light that is incident as A matches our needed initial conditions. Light incident as B, at nearly $90°$ doesn't, and probably won't propagate through the device at all. For incident light in between

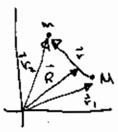
5 these conditions, there will be some that propagate. Now, since $y = y_0 \sin kx$ was a solution for our $n(y)$, it's likely that $y = y_0 \cos kx$ is, too. So, the ~~general~~ path is a little more general even for this k. If the linear combination of sin & cos paths won't match the initial conditions, the propagation can still be sinusoidal. In particular, if the initial conditions match

$$y = y_0' \sin(k'x - \delta')$$

with $y_0' \leq y_0$, then the path will still be sinusoidal.

A

8 a) First, we need to write the Lagrangian:



$$L = \tfrac12 M|\dot{\vec r}_1|^2 + \tfrac12 m|\dot{\vec r}_2|^2 - \tfrac12 \overset{M\omega^2}{\cancel{k}} r_1^2 - \tfrac12 \overset{m\omega^2}{\cancel{k}} r_2^2 - \kappa \frac{Qq}{|\vec r_1 - \vec r_2|}$$

isotropic!

To get to the "one-body" problem, we use CM & rel coordinates $\vec R$ & $\vec r$ defined as

$$\cancel{M\vec R = } \quad (M+m)\vec R = M\vec r_1 + m\vec r_2$$
$$\vec r = \vec r_2 - \vec r_1 .$$

Or,

$$\vec r_1 = \vec R - \frac{m}{M+m}\vec r \qquad \vec r_2 = \vec R + \frac{M}{M+m}\vec r .$$

So, we know the kinetic energy & Coulomb energy ($U_c = \kappa \frac{Qq}{r}$) work out since we've done those before. We thus need only check the harmonic potential:

$$\tfrac12 \overset{M\omega^2}{\cancel{k}} r_1^2 + \tfrac12 \overset{m\omega^2}{\cancel{k}} r_2^2 = \tfrac12 \cancel{k}\Big[\overset{M\omega^2}{}\big(\vec R - \tfrac{m}{M+m}\vec r\big)^2 + \overset{m\omega^2}{}\big(\vec R + \tfrac{M}{M+m}\vec r\big)^2\Big]$$

$$= \tfrac12 \cancel{k}\Big[\Big(R^2 - \tfrac{2m}{M+m}\vec R\cdot\vec r + \big(\tfrac{m}{M+m}\big)^2 r^2\Big)M$$
$$+ m\Big(R^2 + \tfrac{2M}{M+m}\vec R\cdot\vec r + \big(\tfrac{M}{M+m}\big)^2 r^2\Big)\Big]\omega^2$$

$$= \frac12\Big[\underbrace{(M+m)}_{\text{total mass}} R^2 + \underbrace{\frac{Mm}{M+m}}_{\text{reduced mass}} r^2\Big]\omega^2$$

B

So,

$$L = \tfrac12 (M+m)|\dot{\vec R}|^2 - \tfrac12 (M+m) R^2\omega^2 \quad \longleftarrow \text{CM motion}$$
$$+ \tfrac12 \mu |\dot{\vec r}|^2 - \tfrac12 \mu\omega^2 r^2 - \kappa\frac{Qq}{r} . \quad \longleftarrow \text{rel. motion}$$

In fact, we have <u>two</u> equivalent one-body problems to solve to get the general motion.

b) Conserved quantities have $\frac{\partial L}{\partial q} = 0$. So, from our Lagrangian

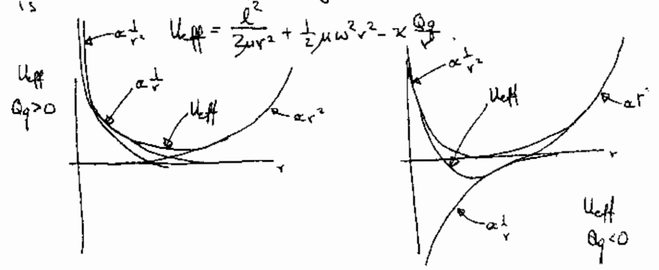
$$\frac{\partial L}{\partial \Theta} = \frac{\partial L}{\partial \Phi} = 0 \quad \longleftarrow \text{ang. mom. of CM motion conserved}$$

$$\frac{\partial L}{\partial \theta} = \frac{\partial L}{\partial \varphi} = 0 \quad \longleftarrow \text{ang. mom. of rel. motion conserved}$$

$$\frac{\partial L}{\partial t} = 0 \quad \longleftarrow \text{energy conserved}$$

c) The effective potential for the rel. motion is

$$U_{eff} = \frac{\overset{\text{rel. ang. mom.}}{\ell^2}}{2\mu r^2} + \tfrac12 \mu\omega^2 r^2 - \kappa \frac{Qq}{r} .$$



C

In both cases, circular orbits are possible, and they will be stable if $\ell > 0$. If $\ell = 0$, stable circular orbits are still possible for $Qq>0$, but not for $Qq<0$. The circular orbits will have a radius determined by

$$\frac{\partial U_{eff}}{\partial r} = 0$$

$$\frac{\partial U_{eff}}{\partial r} = -\frac{\ell^2}{\mu r^3} + \mu\omega^2 r + \kappa\frac{Qq}{r^2} = 0$$

$$\Rightarrow \; -\frac{\ell^2}{\mu} + \mu\omega^2 r_0^4 + \kappa Qq\, r_0 = 0 \qquad r_0 \equiv \text{equilibrium/circ. orbit}$$

We can't solve this easily, but it is the correct condition.

d) The frequencies of small oscillations will be found from

$$U_{eff} \approx \tfrac12 \mu\omega_{eff}^2 (r - r_0)^2 \qquad r_0 \text{ from (c)}$$
$$= \tfrac12 \frac{\partial^2 U_{eff}}{\partial r^2}\Big|_{r_0} (r - r_0)^2$$

or

$$\omega_{eff}^2 = \frac{1}{\mu}\frac{\partial^2 U_{eff}}{\partial r^2}\Big|_{r_0}$$

The corresponding normal mode is simply radial oscillation in the relative coordinate. Note, though,

D

that we also have

$$U_{eff}^{CM} = \frac{L^2}{2(M+m)R^2} + \tfrac12 (M+m) R^2 .$$

So, circular orbits of the CM are also possible, and they will have their own characteristic frequency.

The total motion is thus a superposition of the CM and relative motion.

9. From the discussion on p.3 of the text, a force is conservative if the work ~~to~~ done by the force as a particle moves between two points is independent of the path. In particular, if the particle is returned to its original position, then the net work done must be zero — and path independent. Thus,

$$W_{net} = \oint \vec{F} \cdot d\vec{s} = 0.$$

Stokes' theorem says

$$\oint \vec{F} \cdot d\vec{s} = \int (\vec{\nabla} \times \vec{F}) \cdot d\hat{n}.$$

can be made zero for some special paths, but path independence requires

Thus, the condition $\vec{\nabla} \times \vec{F} = 0$ guarantees that $W_{net} = 0$. Further, if this is true, then we can define

$$\vec{F} = \vec{\nabla} U$$

since

$$\vec{\nabla} \times (-\vec{\nabla} U) \equiv 0.$$

These conditions are <u>not</u> the same as for energy conservation. These must hold — but we also must require $\frac{\partial U}{\partial t} = 0$ for energy conservation.

10. Minimize

$$E[\psi] = \int_a^b dx\, \psi\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V\right]\psi$$

subject to

$$\int_a^b dx\, |\psi|^2 = 1.$$

We include the constraint with a Lagrange multiplier

$$E[\psi] = \int_a^b dx \left\{\psi\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V\right]\psi + \lambda\left(|\psi|^2 - \frac{1}{b-a}\right)\right\}.$$

Also, following the hint, we note that

$$\frac{d}{dx}\left(\psi\frac{d\psi}{dx}\right) = \frac{d\psi}{dx}\frac{d\psi}{dx} + \psi\frac{d^2\psi}{dx^2}$$

so that

$$E[\psi] = \int_a^b dx \left\{-\frac{\hbar^2}{2m}\left(\frac{d}{dx}\left(\psi\frac{d\psi}{dx}\right) - \frac{d\psi}{dx}\frac{d\psi}{dx}\right) + \psi V\psi + \lambda\left(|\psi|^2 - \frac{1}{b-a}\right)\right\}$$

$$= \underbrace{\cancel{-\frac{\hbar^2}{2m}\psi\frac{d\psi}{dx}\Big|_a^b}}_{0} + \int_a^b dx\left[\frac{\hbar^2}{2m}\frac{d\psi}{dx}\frac{d\psi}{dx} + \psi V\psi + \lambda\left(|\psi|^2 - \frac{1}{b-a}\right)\right]$$

if $\psi = 0$ @ a & b or $\frac{d\psi}{dx} = 0$ @ a & b (or a mix); this preserves Hermiticity --

Finally, our functional is

$$E[\psi] = \int_a^b dx \overbrace{\left[\frac{\hbar^2}{2m}\left(\frac{d\psi}{dx}\right)^2 + V\psi^2 + \lambda\left(\psi^2 - \frac{1}{b-a}\right)\right]}^{\equiv\, \mathcal{E}}.$$

The Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{\partial \mathcal{E}}{\partial \psi'}\right) - \frac{\partial \mathcal{E}}{\partial \psi} = 0$$

and this satisfies $\delta E = \frac{\delta E}{\delta \psi} = 0$.

We thus have

$$\frac{d}{dx}\left(\frac{\hbar^2}{m}\psi'\right) - 2V\psi + 2\lambda\psi = 0$$

or

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = +\lambda\psi.$$

This suggests — by comparison with the time-independent Schrödinger equation — that we identify λ as the energy.

Note that an alternative approach is to minimize

$$E[\psi] = \frac{\int_a^b dx\, \psi\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right)\psi}{\int_a^b dx\, |\psi|^2},$$

but with no constraint.