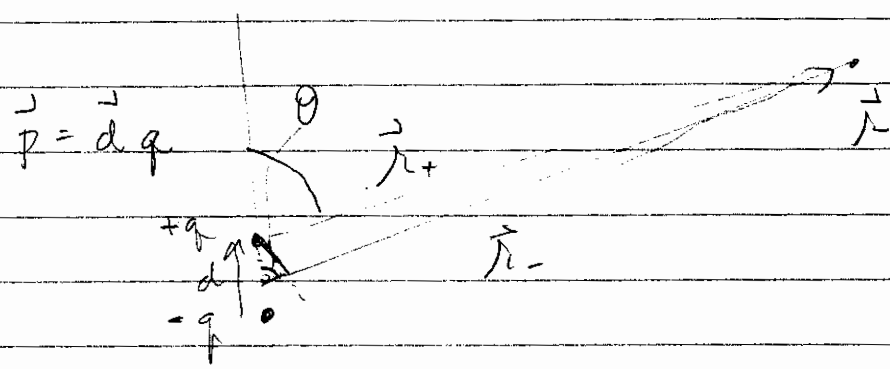
(21)

<u>What is a E dipole?</u>
what is the V, E due to it?

<u>Practice:</u>



V at $(\vec{r})$ is $\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right]$

Clear that it is 0 if $d \to 0$, relies on difference between r^+ & r^-; also, no $1/r$ part.
If $r \gg d$, $r_\pm = r \mp \frac{d}{2}\cos\theta$

So $\frac{1}{r^+} - \frac{1}{r^-} = \frac{1}{r}\left[\underbrace{\frac{1}{1-\frac{d}{2r}\cos\theta} - \frac{1}{1+\frac{d}{2r}\cos\theta}}_{1+\frac{d}{2r}\cos\theta - 1 + \frac{d}{2r}\cos\theta}\right]$

$V = \overbrace{\frac{q}{4\pi\varepsilon_0}\,\frac{d\cos\theta}{r^2}}^{p} = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} \sim \frac{1}{r^2}.$

In general, V (multipole) $\sim \frac{1}{r^{(l+1)}}$.

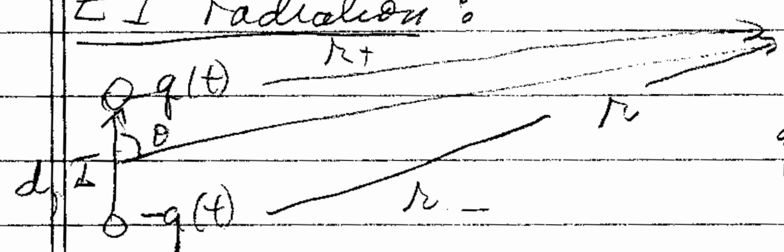
E1 and M1 radiation:

Oscillating charges (Static charges and currents make fields which $\sim 1/r^2$, so $E\times B \sim (1/r^4)$ and $\int (1/r^4)\, r^2\, d\Omega \to 0$ as $r\to\infty$: cannot send forth power)

We seek that part of E & B which $\sim 1/r$ at large r, giving rise to $S \sim 1/r^2$ or a finite power. Charge has to oscillate (or, generally accelerate) to do this.

We pretend we do not know about $E \sim \frac{-q}{4\pi\epsilon_0 c^2}\left[\frac{a_\perp}{R}\right]$ (but will check it later), and use retarded potentials, for practice.

E1 radiation:



$I = I_0 e^{-i\omega t}$

$q(t) = \int I\,dt = \underbrace{\left(\frac{I_0}{-i\omega}\right)}_{q_0} e^{-i\omega t}$

We work out V and A. $\left(p_0 = \left(\frac{I_0}{-i\omega}\right) d\, e^{-i\omega t}\right)$

V: $V = \frac{1}{4\pi\epsilon_0}\int \left[\frac{\rho}{R}\right] d\tau$ (evaluated at $t' = t - R/c$ where $\vec{R}$ is $\vec{r}_+$ or $\vec{r}_-$.)

(Let's assume v of q is $\ll c$, so do not have to include the "Doppler shift" term in $\int \rho\, d\tau$)

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q_0 e^{-i\omega(t - r_+/c)}}{r_+} - \frac{q_0 e^{-i\omega(t - r_-/c)}}{r_-}\right]$$

Note that this is exactly 0 if we neglect the difference between r_+ and r_-, just as we would get 0 for a static dipole if we did this. The juice is in the difference between r_+ and r_-. If $\underline{r >> d}$

$$r_+ = r - d/2 \cos\theta, \quad r_- = r + d/2 \cos\theta$$

$$V = \frac{q_0}{4\pi\varepsilon_0}\left(\frac{e^{-i\omega(t - \frac{r}{c} + \frac{d}{2c}\cos\theta)}}{r\left(1 - \frac{d}{2r}\cos\theta\right)} - \frac{e^{-i\omega(t - \frac{r}{c} - \frac{d}{2c}\cos\theta)}}{r\left(1 + \frac{d}{2r}\cos\theta\right)}\right)$$

We want it to look like some kind of outward traveling wave, so factor out the $e^{i(kr-\omega t)}$, remembering $k = \omega/c$

get lost to lead me through this

$$V = \frac{q_0}{4\pi\varepsilon_0}\frac{e^{i(kr-\omega t)}}{r}\underbrace{\left[\frac{e^{-i\frac{kd}{2}\cos\theta}}{\left(1 - \frac{d}{2r}\cos\theta\right)} - \frac{e^{+i\frac{kd}{2}\cos\theta}}{\left(1 + \frac{d}{2r}\cos\theta\right)}\right]}$$

to first order, these terms cancel. OK, expand to first order in kd and d/r:

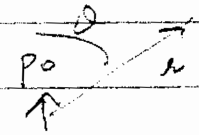
$$\left(1 - ikd/2 \cos\theta\right)\left(1 + \tfrac{d}{2r}\cos\theta\right) - \left(1 + ikd/2 \cos\theta\right)\left(1 - \tfrac{d}{2r}\cos\theta\right)$$

$$V = \frac{\overbrace{q_0 d}^{p_0}}{4\pi\varepsilon_0}\cos\theta\left(-ik + 1/r\right)\frac{e^{i(kr-\omega t)}}{r}$$

where p_0 is the dipole moment $p_0 \equiv q_0 d$

$$V = \frac{p_0}{4\pi\varepsilon_0}\left[\frac{1}{r} - ik\right]\frac{e^{i(kr-\omega t)}}{r}$$

What is V for static dipole p_0?



P172 $V_{dipole} = \dfrac{\vec{p}_0 \cdot \hat{r}}{4\pi\epsilon_0 r^2} = \dfrac{p_0 \cos\theta}{4\pi\epsilon_0 r^2}$

So the $1/r^2$ term is the static part, the $-ik$ the radiation:

$$\boxed{V_{rad} = \frac{p_0\cos\theta}{4\pi\epsilon_0}(-ik)\frac{e^{i(kr-\omega t)}}{r}}$$

($p_0\cos\theta$ is also $\vec{p}_0 \cdot \hat{r}$)

$\vec{A}$: Easier, actually: Only I_z contributes

$$A_z = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{I_0\, e^{-i\omega(t - \frac{r}{c} - \frac{z\cos\theta}{c})}}{(r - z\cos\theta)}\,dz \qquad (r(z) = r - z\cos\theta)$$

where we make the same $r \gg d$ approximation only.

$$= \frac{\mu_0 I_0 e^{i(kr-\omega t)}}{4\pi\, r}\underbrace{\int_{-d/2}^{d/2}\frac{e^{-ikz\cos\theta}}{(1-\frac{z}{r}\cos\theta)}\,dz}$$

$$\int_{-d/2}^{d/2}(1 - ikz\cos\theta)(1 + \tfrac{z}{r}\cos\theta)\,dz$$

OK, this is non-zero to first order.
If $kd \ll 1$ and $d/r \ll 1$, both $\cos\theta$ terms are small,

$$\boxed{A_z = \frac{\mu_0}{4\pi} I_0 d\, \frac{e^{i(kr-\omega t)}}{r}}$$

If we want this in terms of p_0,

$$I_0 d = -i\omega p_0 \qquad \text{so}$$

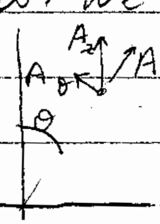
$$\boxed{A_z = \frac{-\mu_0 p_0 (i\omega)}{4\pi} \frac{e^{i(kr-\omega t)}}{r}} \qquad \left(p_0 = \frac{i I_0}{\omega} d \right)$$

(How come Griff. gets no i? Because his $p \sim \cos\omega t$, my $p \sim \sin\omega t$ if I_0 is real. What counts is, A_z is out of phase with p, B comes out <u>in</u> phase with p)

Now get, say,

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta}\right] \hat{\phi}$$

(Oops: we have A_z, not A_θ and A_r:



$$A_r = A_z \cos\theta$$
$$A_\theta = -A_z \sin\theta \;)$$

$$\vec{B} = -\frac{\mu_0 p_0}{4\pi} (i\omega)(-\sin\theta)(ik) \frac{e^{i(kr-\omega t)}}{r} \hat{\phi}$$

$$\boxed{\vec{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi c} \sin\theta \frac{e^{i(kr-\omega t)}}{r} \hat{\phi}}$$

where the A_r term $\sim \frac{1}{r^2}$ and is dropped.

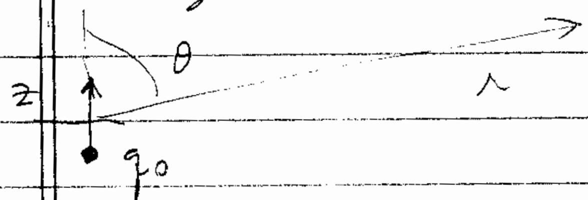
Homework!

$$\boxed{\vec{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi} \left(\frac{\sin\theta}{r}\right) e^{i(kr-\omega t)} \hat{\theta}}$$

Now suppose we *did* know that in radiation region,

$$\vec{E}_{rad} = -\frac{q}{4\pi\epsilon_0 c^2}\left[\frac{a_\perp}{R}\right]_{ret}$$

and make a E1 radiator by oscillating a charge: $v \ll c$



$$z = z_0\, e^{-i\omega t'}$$

$$a_z = -z_0 \omega^2 e^{-i\omega t'}$$

$$p_0 \equiv q_0 z$$

$$a_\perp = a_z \sin\theta = -z_0 \omega^2 e^{-i\omega t'},$$

but evaluate at t'. We are neglecting the "Doppler shift" term. So

$$\vec{E}_{rad} = \frac{-q_0 \sin\theta}{4\pi\epsilon_0 c^2}\left(-z_0\omega^2\right)\frac{e^{-i\omega(t - r/c)}}{r}$$

$$\boxed{\vec{E}_{rad} = -\frac{p_0 \omega^2 \sin\theta}{4\pi\epsilon_0 c^2}\frac{e^{i(kr-\omega t)}}{r}\hat{\theta}}$$

Look familiar?

$$\boxed{\vec{B}_{rad} = \frac{\vec{k}\times\vec{E}}{\omega} = -\frac{p_0 \omega^2 \sin\theta}{4\pi\epsilon_0 c^3}\frac{e^{i(kr-\omega t)}}{r}\hat{\phi}}$$

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Do as quiz

The power from an E1 radiator:

$$\vec{S} = \hat{r}\,\frac{|E||B|}{\mu_0}\cos(\phi)$$

$$= \underbrace{\left(\frac{\mu_0 p_0 \omega^2}{4\pi c}\right)^2 \frac{c}{\mu_0}}\;\underbrace{\cos^2(kr-\omega t)}_{\to\ 1/2}\;\underbrace{\frac{\sin^2\theta}{r^2}}$$

$$\langle \vec{S} \rangle = \hat{r}\left(\qquad\qquad\right)\ \tfrac{1}{2}$$

$$P = \int \langle \vec{S} \rangle \cdot d\vec{a} \ \to\ 2\pi \underbrace{\int_0^\pi \frac{\sin^3\theta\, r^2\, d\theta}{r^2}}_{4/3}$$

$$\boxed{\langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}}$$

Discuss "trick" of p 81

$$= \frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{k\sin^2\theta} \quad \text{or}$$

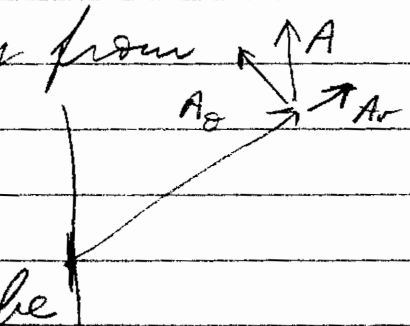
$\lambda/2$ El
$$\boxed{A_z = \frac{\mu_0 I_0 d}{4\pi}\,\frac{e^{i(kr-\omega t)}}{r}\,\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{kd\sin^2\theta}}$$

Compare to

Short El
$$\boxed{A_z = \frac{\mu_0 I_0 d}{4\pi}\,\frac{e^{i(kr-\omega t)}}{r}}$$

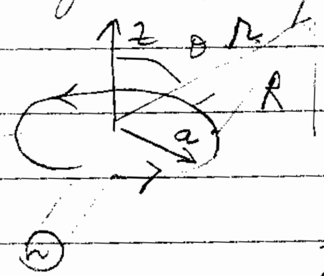
Trick: How to get $\vec{E}$ quickly from $\vec{A}$ in radiation zone:

$$\vec{E} = -\vec{\nabla}V - \partial\vec{A}/\partial t$$



How does $\vec{E}$ come out to be along radial direction? Since $\vec{k}\cdot\vec{E} = 0$ it must. The $\vec{\nabla}V$ term and the $(\partial\vec{A}/\partial t)$ must cancel in the $\hat{r}$ direction. Furthermore, the $\hat{\theta}$ part of $\vec{\nabla}V$ is $\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}$ which $\sim\left(\frac{1}{r^2}\right)e^{i(kr-\omega t)} \to 0$, so dies out as $r\to\infty$ faster than $1/r$. Thus can get $\boxed{E_{rad} \text{ from } (\partial\vec{A}/\partial t)_{\hat{\theta}}}$ part only! Do not even need V at all.

Magnetic dipole (M1):



$I = I_0 e^{-i\omega t}$ $\qquad M = \underbrace{(\pi a^2) I_0}_{M_0} e^{-i\omega t}$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \left[\frac{\vec{J}}{R}\right]_{ret} d\tau = \frac{\mu_0}{4\pi} \int \left[\frac{I}{R}\right] d\vec{l}$$

Place observation point (above) x-axis ($\phi = 0$):

Observation $r_p = (r\sin\theta,\ 0,\ r\cos\theta)$

Source $r_s = (a\cos\phi,\ a\sin\phi,\ 0)$

so

$$R = \sqrt{(r\sin\theta - a\cos\phi)^2 + (a\sin\phi)^2 + (r\cos\theta)^2}$$

$$= r\sqrt{1 - 2\,a/r\sin\theta\cos\phi + a^2/r^2}$$

If $r \gg a$

$$\simeq r\left(1 - \frac{a}{r}\sin\theta\cos\phi\right)$$

so

$$\vec{A} = \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{I_0 e^{-i\omega\left(t - \frac{r}{c} + \frac{a\sin\theta\cos\phi}{c}\right)}}{r(1 - a/r\sin\theta\cos\phi)} a\,d\phi\,\hat{\phi}$$

Which looks easier than it is because $\hat{\phi}$ depends on ϕ: $\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$

So

$$\vec{A} = \frac{\mu_0 I_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \overbrace{\int_0^{2\pi} \frac{e^{-ika\sin\theta\cos\phi}}{(1 - a/r\sin\theta\cos\phi)} a\,d\phi\,\hat{\phi}}$$

$$\int_0^{2\pi} \frac{(1 + ika\cos\phi\sin\theta)(1 + a/r\cos\phi\sin\theta)}{1 + \cos\phi\sin\theta\,[ika + a/r]} \times (\hat{y}\cos\phi - \hat{x}\sin\phi)\, a\,d\phi$$

Note that $\int_0^{2\pi} \sin\phi$ or $\cos\phi \, d\phi \to 0$,
so "I" goes out. Similarly,

$$\int_0^{2\pi} \cos\phi \sin\phi \, d\phi \to 0, \text{ so}$$

only survivor is $\cos^2\phi$

$$\int_0^{2\pi} \cos^2\phi \, d\phi = \pi$$

$$\vec{A} = \hat{y} \, \frac{\mu_0 \overbrace{I_0 a\pi}^{M_z}}{4\pi} \left[ik + \overset{\text{static}}{1/r}\right] \sin\theta \, \frac{e^{i(kr-\omega t)}}{r}$$

The static dipole part is

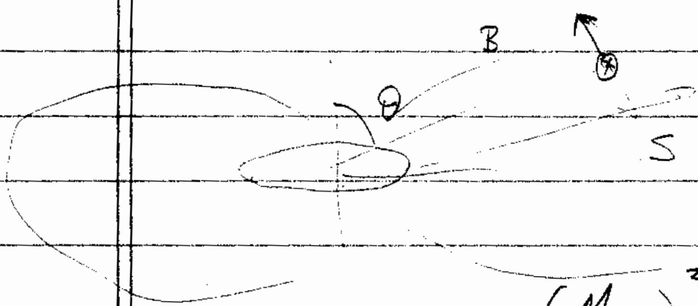
$$\vec{A}_{stat} = \hat{y} \, \frac{\mu_0 M_0}{4\pi} \, \frac{\sin\theta}{r^2} \, e^{i(kr-\omega t)}$$

$$\vec{A}_{rad} = \hat{y} \, \frac{\mu_0 M_0 (ik)}{4\pi} \, \frac{\sin\theta}{r} \, e^{i(kr-\omega t)}$$

Taking $\vec{\nabla} \times \vec{A}$,

$$\vec{B} = - \frac{\mu_0 M_0}{4\pi} \, k^2 \sin\theta \, \frac{e^{i(kr-\omega t)}}{r} \, \hat{\theta}$$

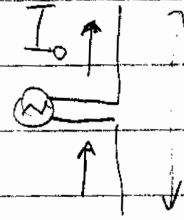
$$\vec{E} = + \frac{\mu_0 M_0}{4\pi} \, k^2 c \, \sin\theta \, \frac{e^{i(kr-\omega t)}}{r} \, \hat{\phi}$$



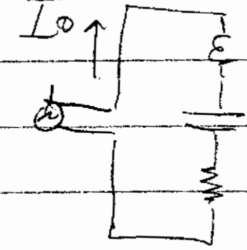
E1 M1 as

$$P_M / P_E = \left(\frac{M_0}{p_0 c}\right)^2 \quad \begin{pmatrix} E \to Bc & p_0 \to M_0/c \\ Bc \to E & \end{pmatrix}$$

Radiation resistance:



looks like



So can define R_{rad} by

$$\langle I_0^2 \rangle R_{rad} = \langle P \rangle$$

$$R_{rad} = \frac{\langle P \rangle}{I_0^2/2} = \mu_0 \overbrace{\frac{I_0^2 d^2}{\omega^2}}^{p_0^2} \frac{\omega^4}{12\pi c} \frac{1}{I_0^2/2}$$

$$R_{rad}^{E1} = (20\,\Omega)(kd)^2$$

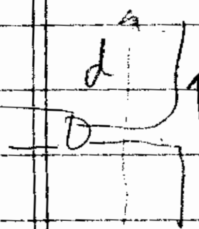
There is an equivalent expression for R_{rad}^{M1}. Do it in problem set.

Observation!

1) In Radiation zone $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = c\vec{B} \times \hat{R}$, we do not really need Φ! Why?
$\vec{\nabla}\phi$ has $\hat{\theta}$ and $\hat{r}$ part, but when add them, only $(-\partial A/\partial t)_{\theta}$ survives!

2) We assumed $t' = t - \frac{R}{c} \simeq t - r/c$, i.e., $kd \ll 1$; if not true, get higher multipoles.

Real antenna:

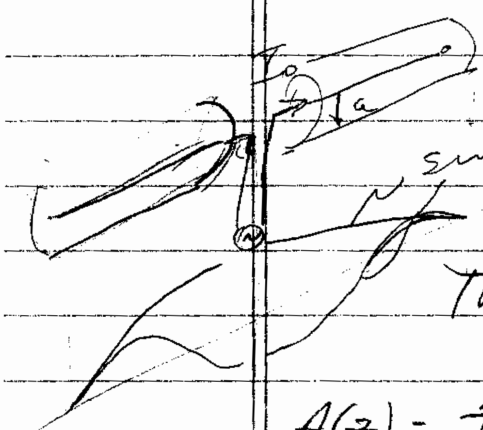


$I = I_0 e^{-i\omega t}$

what really is $I_0(z)$? not constant.

Look at co-ax: $I = I_0^+ e^{i(kz-\omega t)} + I_0 e^{i(-kz-\omega t)}$

such that $I = 0$ at ends,



or $I = I_0 \sin(kz) e^{-i\omega t}$

This is true even as $a \to \infty$. so

$$A(z) = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} I_0 \underbrace{\frac{e^{+i(kr-\omega t)}}{r}}_{\text{take out}} \frac{e^{-ikz\cos\theta}}{(1-\frac{z}{r}\cos\theta)} \underbrace{\sin(kl-k|z|)}\,dz$$

From $\sin(kz)$ relative to end of antenna.

$$= \frac{\mu_0 I_0 e^{i(kr-\omega t)}}{4\pi\, r}\int_{-d/2}^{d/2} e^{-ikz\cos\theta}\,\underbrace{\sin(kl-k|z|)}\,dz$$

This was missing before.

If $kd/2 = \pi/2$

$$\underbrace{\int_{-d/2}^{d/2} e^{-ikz\cos\theta}\cos(kz)\,dz}$$

$$= \int_{-d/2}^{d/2} e^{-ikz\cos\theta}\left(\frac{e^{ikz}+e^{-ikz}}{2}\right)dz$$

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So

$\lambda/2$ El $\quad E_\theta = \frac{\mu_0 I_0 d}{4\pi}(-i\omega)\frac{e^{i(kr-\omega t)}}{r}\boxed{\frac{2\cos(\frac{\pi}{2}\cos\theta)\,\cancel{\sin\theta}}{(\pi\ \sin^{\cancel{2}}\theta)}}$

short El $\quad E_\theta = \frac{\mu_0 I_0 d}{4\pi}(-i\omega)\frac{e^{i(kr-\omega t)}}{r}\boxed{\sin\theta}$

This replaces this.

Can show $\langle P\rangle_{\lambda/2\,El} = (73.1\,\Omega)\langle I_0^2/2\rangle$

instead of $\langle P\rangle_{short\,El} = \underbrace{(20\,\Omega)\,(kd)^2}\langle I_0^2/2\rangle$

$(= 197\,\Omega$ for $kd = \pi)$

Receiving antennae: