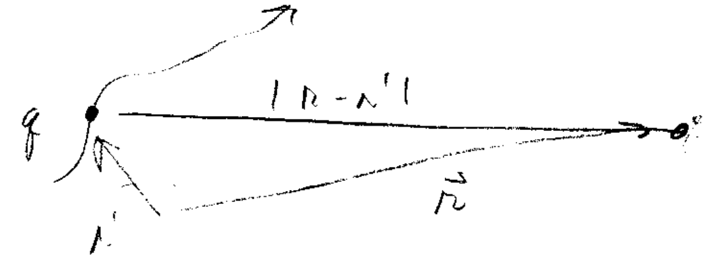
L-W potentials:

Potentials due to moving point charge



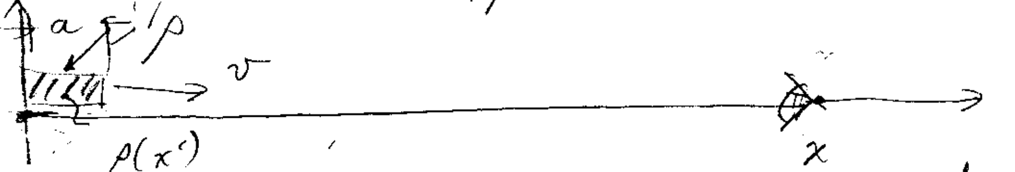
Would expect $V = \frac{1}{4\pi\epsilon_0} \frac{q}{|r - r'|_{t' = t - \frac{|r-r'|}{c}}}$

Note $|r - r'|$ is to be evaluated at the retarded "position" of q.

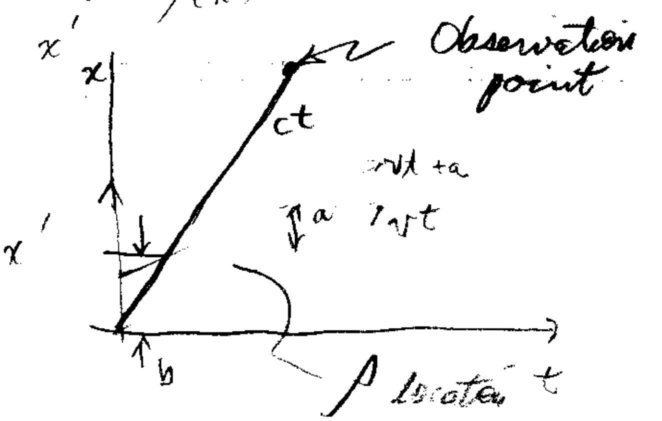
But this is <u>wrong</u> anyway!

Why? $V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r', t' = t - \frac{|r-r'|}{c})}{|r' - r|} \, dr'$

at any time t, the $|r' - r|$ is pretty much localised at the retarded position, but the the $\rho(r', t')\,dr'$ is to be done at fixed $\underline{t}$, not fixed $\underline{t}'$, which means that, if the charge is moving, there is a kind of "Doppler shift" effect. Do it in 1-D!



observer at x, t.



Let $x = ct$ define origin of coord system, charge at $x' = 0$

(6.1)

Integral is $V = \frac{1}{4\pi\varepsilon_0} \frac{\int \rho(x'; t' = t - \frac{x-x'}{c})\, dx'}{|x-x'|}$

ρ is zero from 0 to $\underline{b}$, not 0 to $\underline{a}$.
How are a & b related? $b = c\left(\frac{b-a}{v}\right)$ or
$$b(1 - c/v) = -c\, a/v$$
or $b/a = 1/(1 - v/c)$

so

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \frac{\rho b}{(x-x')} = \frac{q}{4\pi\varepsilon_0} \frac{1}{|x-x'|(1 - v/c)}}$$

That is, because charge is moving toward you, the light leaving the front is a bit closer to you than that leaving the back by $(1 - v/c)^{-1}$ (the train is "louder").
What if we do it in 2D? Only the velocity component pointing toward or away enters in this shift, so

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \frac{q}{|r - r'|} \frac{1}{\left(1 - \underbrace{\vec{v}\cdot\left(\frac{\hat{r-r'}}{c}\right)}_{\vec{v}(t') \text{ along } (\vec{r} - \vec{r}')}\right)}}$$ or

Text uses $\vec{r} - \vec{r}' = r$ bold! Ugh.
It's messy to have both r' & r in problems, but we can usually eliminate one or other.
Similarly, since $\vec{j} = \rho\vec{v}$, exactly the same analyses gives

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi} \frac{q\, \vec{v}(t' = t - \frac{|r'-r|}{c})}{|r'-r|\,\left(1 - \vec{v}(t')\cdot\frac{(\hat{r-r'})}{c}\right)} = \frac{\vec{v}}{c^2} V}$$

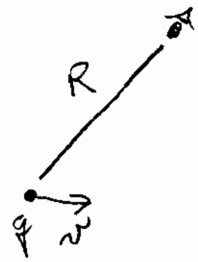
The $\vec{v}$ is at the retarded time.

or more compact: let $\vec{R} = \vec{r} - \vec{r}\,'$

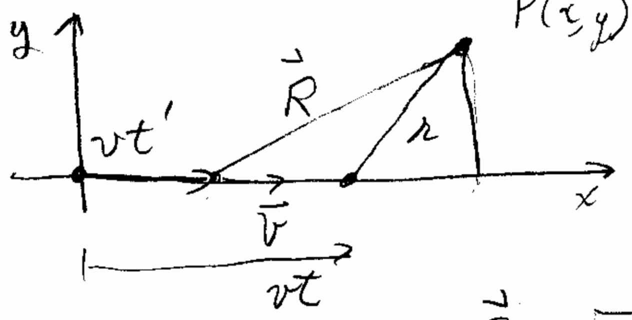
$$V = \frac{q}{4\pi\varepsilon_0} \frac{1}{\left[R - \frac{\vec{v}}{c}\cdot\vec{R}\right]_{Ret}}$$

Retarded

$$A = \frac{[\vec{v}]}{c^2} V$$



Potential due to moving charge at constant $\vec{v}$:



$$t' = t - \vec{R}/c$$

$$\vec{R} = \sqrt{(x - vt)^2 + y^2} \quad \text{(Leave out } z\text{)}$$

We need $R(t) = \sqrt{(x - v(t + R/c))^2 + y^2}$

This is messy, but straightforward. Expand it out and use quadratic formula, then put R into eq'n for V, using $\vec{v}\cdot\vec{R}' = R/(x - vt')$, which also has to be evaluated at t. There is a lot of (simple but messy) algebra to yield

$$\boxed{V = \frac{q}{4\pi\varepsilon_0} \frac{\gamma}{\left[\gamma^2 (x - vt)^2 + y^2 + z^2\right]^{1/2}} \quad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}}$$

$$\boxed{\vec{A} = \frac{\vec{v}}{c^2} V}$$

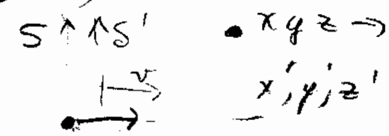
(OK, added the z back in)

Note: If $v=0$,

$$V = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{(x^2+y^2+z^2)}$$

$$\vec{A} = 0 \quad \text{as expected.}$$

If move at $\vec{v}$ relative to charge, let



$$\begin{array}{lll} S & & S' \\ x & \to & \gamma(x - vt) \\ y & \to & y \\ z & \to & z \\ V & \to & V = \dfrac{q}{4\pi\varepsilon_0}\,\gamma\,\dfrac{1}{\sqrt{\gamma^2(x-vt)^2+y^2+z^2}} \\ A & \to & \dfrac{\vec{v}}{c^2}V \end{array}$$

? Part of 4-vector transf.

What are fields? (Call $R' \equiv \sqrt{\gamma^2(x-vt)^2+y^2+z^2}$,

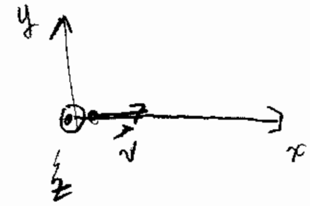
Not fundamental, just convenience for doing the algebra)

$$\boxed{\begin{aligned} V &= \frac{q}{4\pi\varepsilon_0}\,\frac{\gamma}{R'} \\ \vec{A} &= \frac{\vec{v}}{c^2}V \end{aligned}}$$

B is easy: $\vec{B} = \vec{\nabla}\times\vec{A} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ vV/c^2 & 0 & 0 \end{pmatrix}$

$$= \frac{qv\gamma}{4\pi\varepsilon_0 c^2}\underbrace{\left(\hat{y}\frac{\partial}{\partial z} - \hat{z}\frac{\partial}{\partial y}\right)\left(\frac{1}{R'}\right)}_{\left(\frac{-1}{R'}\right)^2\left(\frac{1}{R'}\right)\left(\frac{2z\hat{y} - 2y\hat{z}}{2}\right)}$$

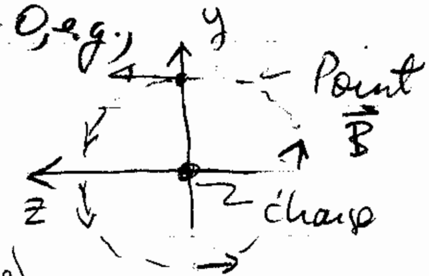
(6.4)



$$\boxed{B_y = \frac{q v \gamma}{4\pi\varepsilon_0 c^2}\left(\frac{1}{R'^3}\right)(-z)}$$

$$\boxed{B_z = \quad '' \qquad (+y)}$$

Which way does it go? Let $z = 0$, e.g.,



$$\boxed{B_z = \frac{q v \gamma}{4\pi\varepsilon_0 c^2}\left(\frac{1}{R'}\right)^3 y}$$

(Rather like "$\dfrac{i\,d\vec{l} \times \vec{R}}{R^3}$" $\left(\dfrac{\mu_0}{4\pi}\right)$

and in fact $i\,d\vec{l} \to q\vec{v}$, get Biot-Savart law)

Electric Field:

$$E_x = -\nabla V - \partial \vec{A}/\partial t$$

$$E_x = \frac{q}{4\pi\varepsilon_0}\left[\frac{\gamma}{(R')^3}(\gamma^2(x - vt)) + \underbrace{\frac{v}{c^2}\frac{\gamma}{(R')^3}\gamma^2(x - vt)(-v)}_{-\frac{\partial A_x}{\partial t}}\right]$$

$$\boxed{E_y = \frac{q}{4\pi\varepsilon_0}\frac{\gamma}{(R')^3}\, y}$$

$$\boxed{E_z = \underbrace{\frac{q}{4\pi\varepsilon_0}\frac{\gamma}{(R')^3}\, z}_{-\nabla V}}$$

$$\boxed{E_x = \frac{q\gamma}{4\pi\varepsilon_0 (R')^3}(x - vt)}$$

$$\gamma^2(1 - v^2/c^2) = 1$$

~~Note that~~ $\cancel{\frac{\vec{v} \times \vec{E}}{c^2} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v & 0 & 0 \\ E_x & E_y & E_z \end{vmatrix} = \frac{1}{c^2} = \hat{1}_z\, v B_z}$

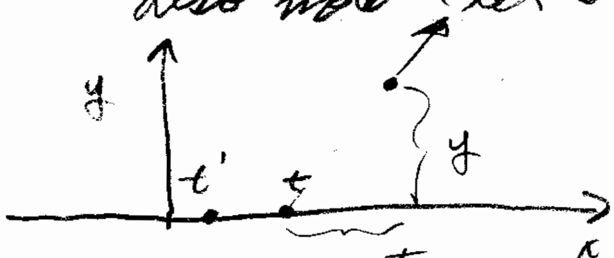
$$\begin{vmatrix} \hat x & \hat y & \hat z \\ v & 0 & 0 \\ E_x & E_y & E_z \end{vmatrix}$$

Note that $B_y = -\frac{v}{c^2} E_z$, $B_z = \frac{v}{c^2} E_y$, $\vec{v} = \hat{x} v$

or $\boxed{\vec{B} = \frac{\vec{v} \times \vec{E}}{c^2}}$ Hmmm---.

We know that if you move at $\vec{v}$ through $\vec{B}$ you get $\boxed{\vec{E} = \vec{v} \times \vec{B}}$ apparently it is also true that if you move at $\vec{v}$ through $\vec{E}$ you get $\vec{B} = \frac{\vec{v} \times \vec{E}}{c^2}$, where the c^2 is just there to make the units right!

also note (let $z=0$) that

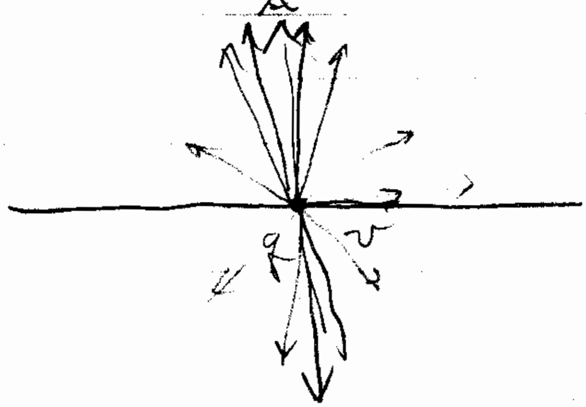


$$\frac{E_y}{E_x} = \left(\frac{y}{x - vt}\right)$$

So $\vec{E}$ points directly away from present location of charge! (Cute).

also, at $x - vt = 0$, $E_y = \underbrace{\frac{q}{4\pi\epsilon_0} \frac{1}{y^2}}_{\text{Normal Coulomb field}} \underbrace{\gamma}_{\text{amplifier}}$

($R' = y$ at this time)



Field lines "squunch up" $\perp$ to motion of charge.

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accelerated point charge:

General case, with charge in non-uniform motion: There are many derivations and many final expressions, including Feynman Vol I

$$\vec{E} = -\frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{R}}{R^2} + \frac{R}{c}\frac{d}{dt}\left(\frac{\hat{R}}{R^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\hat{R}\right]$$

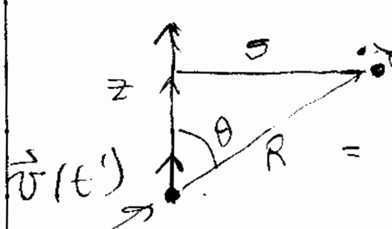
where $\vec{R}$ is $(r'-r)$ evaluated at retarded time.

I am sure this is the same as Griffiths 10.65, but nobody will ever know.

In non-rel. limit, there is an easier way: Get $\vec{A}$, then get $\vec{B}$, then get $\vec{E}$!

$$\vec{A}(t) = \frac{\mu_0}{4\pi}\left[\frac{\vec{v}}{R}\right]$$ where R is retarded position and $\vec{v}$ is retarded time.

$$t' = t - R/c$$



eye, observer at (s, z)

$R = \sqrt{s^2+z^2}$

Let $\vec{v}$ lie along z, but can be changing.

charge $\vec{v}(t')$ is velocity of charge. Just plug in, get $\vec{A}$. But let's get $\vec{B}$, which is $\nabla\times A$, and looks easy (but is tricky).

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\Big(\frac{1}{s}\underbrace{\frac{\partial A_z}{\partial\phi}}_{\to 0}\hat{s} - \frac{\partial A_z}{\partial s}\hat{\phi}\Big) = -\hat{\phi}\frac{\partial}{\partial s}\, q\frac{\vec{v}(t')}{R}\left(\frac{\mu_0}{4\pi}\right)$$

$$B_\phi = -\frac{\mu_0 q}{4\pi}\frac{\partial}{\partial s}\left(v\Big(t - \frac{\sqrt{s^2+z^2}}{c}\Big)(s^2+z^2)^{-1/2}\right)$$

$$\left[\dot{v}(t')\frac{(s^2+z^2)^{-1/2}(2s)}{2c}(s^2+z^2)^{-1/2} + v\,(s^2+z^2)^{-3/2}(-s)\right]$$

$$B_\phi = +\frac{\mu_0 q}{4\pi}\left(\frac{v\,R\sin\theta}{R^3} + \frac{\dot{v}}{c}\frac{R\sin\theta}{R^2}\right) \qquad s = R\sin\theta)$$

where v, R are evaluated at retarded time,
or since $\hat{z} \times \hat{R} = +\hat{\phi}\, R \sin\theta$

$$\boxed{\vec{B} = +\frac{\mu_0 q}{4\pi}\left[\frac{\vec{v}}{R^3} + \frac{\dot{\vec{v}}}{cR^2}\right] \times \vec{R}}$$

What if $\vec{v}$ is not along $\dot{\vec{v}}$ (which we make along z)? We just add a second piece due to $v_\perp$, which does not contribute to $\dot{\vec{v}}$ term, so $\vec{v}$ term is

$$\frac{\vec{v}}{R^3} \times \vec{R} \quad \text{becomes} \quad \frac{\vec{v}_z \times \vec{R}}{R^3} + \frac{\vec{v}_s \times \vec{R}}{R^3} = \frac{\vec{v} \times \vec{R}}{R^3},$$

so above result does not require $v \parallel$ to $\dot{v}$.
(True only if $v \ll c$, all of this).

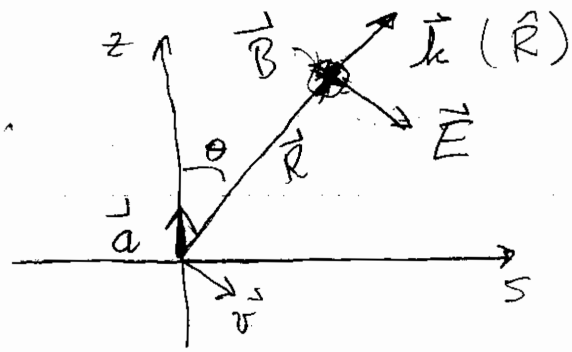
What are these terms? The $\vec{B} = \frac{\mu_0 q}{4\pi} \frac{\vec{v} \times \vec{R}}{R^3}$ is the Biot-Savart law, except that $\vec{v}$ and R^3 have to be evaluated at retarded time. Dies as $1/R^2$.

The $\boxed{\vec{B} = \frac{\mu_0 q}{4\pi c}\left[\frac{\vec{a} \times \vec{R}}{R^2}\right]}$ is the radiation field, dies as $\underline{\underline{1/R}}$.

Now to get $\vec{E}$ we can
1) Make an argument, or 2) Grind out $-\vec{\nabla}\phi - \partial \vec{A}/\partial t$

The argument is that any time dependent wave in free space can be represented by a Fourier sum of plane waves locally, and for each of these $i\vec{k} \times \vec{B} = -i\frac{\omega}{c^2}\vec{E}$ where $\hat{k} = \hat{R}$, direction of radiation.

So picture is:



$B \sim B_\phi$

$E \sim E_\theta \qquad |\vec E| = \frac{k c^2}{\omega}|\vec B| = c|\vec B|$

In terms of $\vec a$ and θ,

$$|\vec E|_{rad} = c\frac{\mu_0 q}{4\pi}\frac{aR}{cR^2}\sin\theta$$

$$\boxed{\begin{gathered}\vec E_{rad} = \frac{q}{4\pi\varepsilon_0 c^2}\left[\frac{a\sin\theta}{R}\right]\hat\theta \\ \vec E_{rad} = \frac{-q}{4\pi\varepsilon_0 c^2}\left[\frac{\overrightarrow{\text{Retarded transverse accel.}}}{R}\right]\end{gathered}}$$

What power is in this wave?

$$\vec S = \frac{\vec E\times\vec B}{\mu_0} = \overset{B}{\frac{\cancel{\mu_0} q}{\cancel{\mu_0} 4\pi c}}\cdot\overset{E}{\frac{q}{4\pi\varepsilon_0 c^2}}\frac{a^2}{R^2}\sin^2\theta\ \hat R$$

$$\vec S =_c \sqrt{\frac{\mu_0}{\varepsilon_0}}\left(\frac{q\, a\sin\theta}{4\pi R}\right)^2\hat R$$

Integrate over any [R]

$$P = \int \vec S R^2\, d\Omega\, dR$$

$$= \frac{1}{c^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left(\frac{qa}{4\pi}\right)^2 2\pi\int_0^\pi \underbrace{\sin^3\theta\, d\theta}_{4/3}$$

$$= \frac{1}{c}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{(qa)^2}{4\pi}\frac{2}{3}$$

$$P = \mu_0\frac{q^2[a]^2}{4\pi}\frac{2}{3}$$

This power arrives at R/c after [a].

Recap of Potentials & Fields: $\vec A$ & V:

$\vec B = \vec\nabla\times\vec A$, $E = -\nabla V - \partial\vec A/\partial t$

Maxwell's eq'ns become wave eq'n

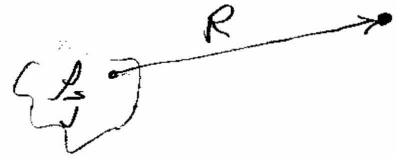
$$\left(-\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix}V\\ A\end{Bmatrix} = \begin{Bmatrix}\rho/\varepsilon_0\\ \vec J\mu_0\end{Bmatrix}$$

for which sol'n is

$$V(t) = \frac{1}{4\pi\varepsilon_0}\int\left[\frac{\rho}{R}\right]_{ret} d\tau$$

$$\vec A(t)\ \frac{\mu_0}{4\pi}\int\left[\frac{\vec J}{R}\right]_{ret} d\tau$$

where ρ and R are evaluated at retarded time



Ex: Problems, Examples

From this we derive:

L-W potentials for single charge in $\vec v(t)$,

deriving $V(t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R - \vec R\cdot\vec v/c}\right]$, where R and $\vec v$ are at retarded t.

$\vec A = \frac{[\vec v]}{c^2}V$ where $[\vec v]$ is retarded.

for constant $\vec v = \hat x v$, these

yield $V = \frac{q}{4\pi\varepsilon_0\underbrace{(\gamma^2(x-vt)^2+y^2+z^2)^{1/2}}_{\equiv R'}}$, $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

$\vec A = \frac{\vec v}{c^2}V$

$$\left\{\begin{array}{l} \vec E_{x,y,z} = \begin{Bmatrix}\hat x\\ \hat y\\ \hat z\end{Bmatrix}\dfrac{q}{4\pi\varepsilon_0(R')^3}\begin{Bmatrix}(x-vt)\\ y\\ z\end{Bmatrix} \\ \vec B = \dfrac{\vec v\times\vec E}{c} \end{array}\right.$$

where the explicit t dependence is built in.

The general expression, if $\vec{v}$ is time dependent,

$$\vec{B}(t) = \frac{\mu_0 q}{4\pi}\left[\left(\frac{\vec{v}}{R^3} + \frac{\vec{a}}{cR^2} \times \vec{R}\right)\right] \quad (\text{if } v \ll c \text{ only})$$

still evaluating at retarded time.

The $\vec{E}$ is messy, but as $R \to \infty$ only $\vec{B}_{rad}$ survives (the $\vec{a}$ term), and an argument leads to

$$\boxed{\vec{E}(t)_{rad} = -\frac{q}{4\pi\varepsilon_0}\left[\frac{a_\perp}{R}\right]}, \quad v \ll c.$$

(An exact analysis shows

$$\vec{E}(t)_{rad} = -\frac{q}{4\pi\varepsilon_0}\frac{d^2}{dt^2}[\hat{R}] \qquad \text{any } v,$$

which is $\approx \left[\frac{1}{R}\right]\frac{d^2}{dt^2}[x_\perp]$, not

$$\frac{d^2}{dt'^2}[x_\perp]$$
)

The above eq'n is very useful and intuitive since it associates $\vec{E} \sim [a_\perp]$ one to one.

$$\boxed{\vec{B}_{rad}(t) = \frac{\tilde{k} \times \vec{E}}{\omega}} \quad \text{from Faraday's Law.}$$