(5)

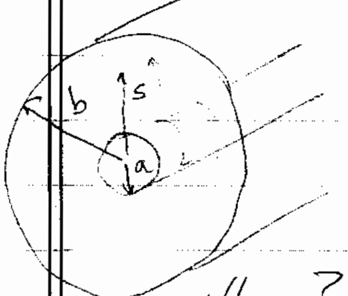
Co-ax. wave guide: for TE/TM guide, no TEM mode (see. p 407): Must have transverse wave. If coax, have central conductor, now is possible.

* $B_z = 0 \Rightarrow \partial E^0_y/\partial x - \partial E^0_x/\partial y = 0$ from Faraday's law,
So $\vec{E}^0(x,y) = -\nabla\phi$. Also

$E_z = 0 \Rightarrow \vec{\nabla}\cdot\vec{E} = 0$ $\quad \partial E^0_x/\partial x + \partial E^0_y/\partial y = 0$ or $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$

No charges. If $\phi = 0$ on surface $\Rightarrow \phi = $ const.

Look for $E_z = 0$, $B_z = 0$; at 10^{10} Hz, $\lambda = 3$ cm ($k_t \sim 1/3$cm)



would need at least ~ cm size guide if had k_t of this size.
So need $k_t = 0$ mode, one with only transverse E & B.

How? Use $E = \nabla_t \phi$ in LaPlace's eq'n:
Put it in cylindrical co:ord: But wait, it must look like a charged rod, for which

$2\pi s l E = \lambda l/\varepsilon_0$, $\quad \vec{E} = E_0 a/s \ \hat{s} \leftarrow E_0(x,y)$

So wave down cavity

$$\vec{E} = \hat{s}\, E_0 a\, e^{i(kz-\omega t)}/s$$

$$i\omega\vec{B} = -\vec{\nabla}\times\vec{E} = \hat{\phi}\frac{\partial E_s}{\partial z} - \hat{z}\frac{1}{s}\frac{\partial E_s}{\partial\phi} = -\hat{\phi}\, ik\, E_0 a$$

$$\vec{B} = -\hat{\phi}\,\frac{k}{\omega}\, E_0 a\, e^{i(kz-\omega t)}/s$$

(How come $\vec{\nabla}\times\vec{E} \neq 0$? We claimed it was! No, only $\vec{\nabla}\times\vec{E}^0 = 0$.)

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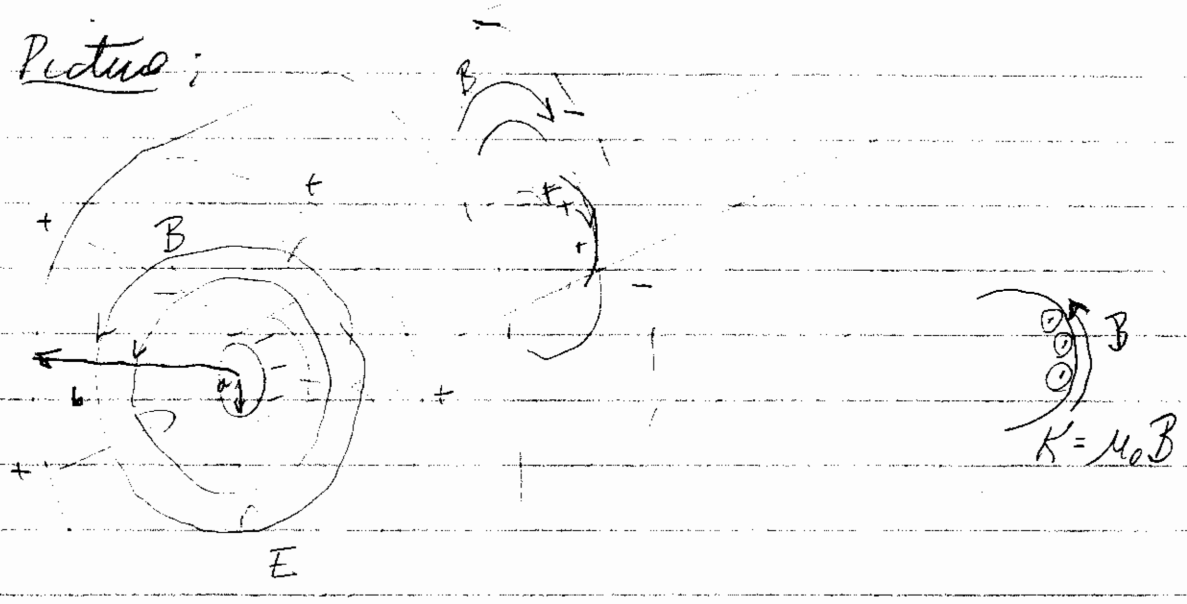
Picture:



The Power:

$$S = \hat{z}\left(\frac{B_0^2}{\mu_0}\right) c \left(\frac{a^2}{s^2}\right) \text{ etc.}$$

"Impedance" of cable:

Apply $V = V_0 e^{i(kz-\omega t)}$, get $I = I_0 e^{i(kz-\omega t)}$

How are V_0 and I_0 related?

$$V = \int_a^b (E_0 a)\, ds/s = E_0 a \ln(b/a)\, e^{i(\sim)}$$

$$I = K\, 2\pi a = 2\pi a \frac{1}{\mu_0} \frac{k}{\omega} E_0 \cancel{a}/\cancel{a}\, e^{i(\sim)}$$

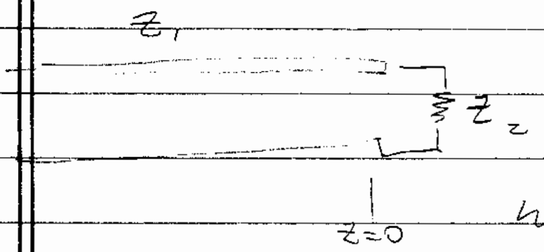
$$\text{Define } Z_{cable} = V/I = \frac{E_0 a \mu_0}{k/\omega\, E_0} \frac{1}{2\pi a} \ln(b/a)$$

$$= \frac{1}{2\pi} \frac{\mu_0 \omega}{k} \ln(b/a) \qquad \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \ln(b/a) = 60 \ln(b/a)\ \Omega.$$

If $b = 25\,mm$, $a = 2\,mm$, $\qquad = \underline{130\ \Omega}$.

What good is this concept?



Send $V_0^+ e^{i(kz-\omega t)}$

reflects $V_0^- e^{i(-kz-\omega t)}$

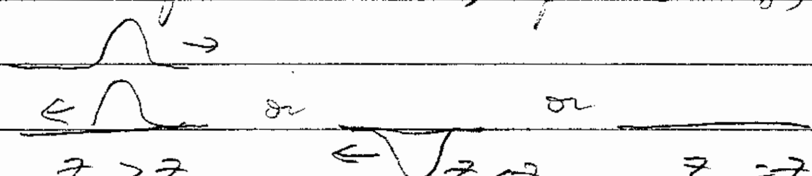
with $I_0^+ = I_0 e^{i(kz-\omega t)}$
$I_0^- = -I_0 e^{i(-kz-\omega t)}$

Since the $\bar{E}$'s add, the V's add, the I's add,

at Z_2 $V/I = Z_2$ or

$$\frac{(V_0^+ + V_0^-)}{(I_0^+ - I_0^-)} = Z_2 = Z_1 \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \qquad (Z_1 = Z_{cable} = V_0/I_0)$$

or $V_0^-/V_0^+ = \dfrac{Z_2 - Z_1}{Z_2 + Z_1}$ If $Z_2 = Z_1$, no reflected wave; if Z_2 large, V reflects in phase



$Z_2 > Z_1$ $\qquad$ $Z_2 < Z_1$ $\qquad$ $Z_2 = Z_1$

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chapt. 10: Potentials & Fields:

Maxwell's eq'ns are simpler in terms of potentials $\vec{A}$ and ϕ

i) $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$ iii

ii) $\vec{\nabla}\cdot\vec{B} = 0$ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\,\partial\vec{E}/\partial t$ iv

Because $\vec{\nabla}\cdot\vec{B} = 0$, $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ some vector field.

Put this into Faraday's Law

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) \quad \text{or}$$

$$\vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0 \quad \text{or} \quad \vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V$$

$$\text{or} \quad \boxed{\vec{E} = -\nabla V - \partial\vec{A}/\partial t}$$

OK, can define a V and $\vec{A}$ which are related to $\vec{E}$ & $\vec{B}$ this way. So now write M.E. in terms of V and $\vec{A}$:

$$\vec{\nabla}\cdot\vec{E} = \boxed{-\vec{\nabla}^2 V - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \rho/\varepsilon_0}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} + \mu_0\varepsilon_0(-)\vec{\nabla}\left(\frac{\partial V}{\partial t}\right) + \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$
$\nabla(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2 A$

$$\boxed{-\nabla^2 A - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} + \vec{\nabla}\left[(\vec{\nabla}\cdot\vec{A}) + \mu_0\varepsilon_0\left(\frac{\partial V}{\partial t}\right)\right] = \mu_0\vec{J}}$$

which are quite ugly.

Gauge Transf: are V and $\vec{A}$ unique? No.

If we have a V and $\vec{A}$, we can use any scalar function λ to transform to V' and $\vec{A}'$ which give same $\vec{E}$ & $\vec{B}$ following the prescription

$$\boxed{\begin{array}{l} \vec{A}' \to \vec{A} + \vec{\nabla}\lambda \\ V' \to V - \partial\lambda/\partial t \end{array}}$$

Why? $\vec{B} = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A}$ (since $\vec{\nabla} \times (\vec{\nabla}\lambda) = 0$) ✓

$$\vec{E} = -\nabla V - \cancel{\vec{\nabla}\frac{\partial\lambda}{\partial t}} + \frac{\partial A}{\partial t} + \frac{\partial(\vec{\nabla}\lambda)}{\partial t} = -\nabla V + \frac{\partial \vec{A}}{\partial t}$$ ✓

Use these to adjust $\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\lambda$ for any scalar to be anything you want. In magnetostatics we chose $\vec{\nabla}\cdot\vec{A}' = 0$. For example, if $\vec{\nabla}\cdot\vec{A} = f(\vec{r})$ $\vec{\nabla}\cdot\vec{A}' = f(r) + \nabla^2\lambda$

So just need to solve $\nabla^2\lambda = -f(r)$ which we can always do. Called "Coulomb gauge".

But this is not convenient here. Instead let's make $\boxed{\vec{\nabla}\cdot\vec{A}' - \mu_0\varepsilon_0\frac{\partial V'}{\partial t} = 0}$ which means

$f(\vec{r}) \equiv \left(\vec{\nabla}\cdot\vec{A} - \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\nabla^2\lambda + \mu_0\varepsilon_0\frac{\partial^2\lambda}{\partial t^2}$, which means solving some scalar wave eq'n. with $f(r)$ as a source. Called "Lorentz" gauge. When V is not time dependent, $\vec{\nabla}\cdot\vec{A} = 0$, same as before.

We do not have to really do this, since just assuming it is possible is sufficient to enable us to find $\vec{A}$ and Φ from the wave equation, given that this condition holds.

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So in Lorentz Gauge, M.E. become

$$\boxed{\left.\begin{aligned} &+\nabla^2 V + \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2} V = -\rho/\varepsilon_0 \\ &\nabla^2 \vec{A} + \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \end{aligned}\right\} \text{and}}$$

4 wave eq'ns, with $\vec{J}$ and ρ as source terms.

These are familiar: If the time derivatives are zero,

$$\nabla^2 V = -\rho/\varepsilon_0 \quad \text{for which}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')}{|r-r'|} d\tau'$$

and $\nabla^2 \vec{A} = -\vec{J}/\varepsilon_0$ or

$$A_x(r) = \frac{\mu_0}{4\pi} \int \frac{J_x(r)}{|r-r'|} d\tau', \text{ sim } y \,\&\, z.$$

What if the time derivatives are not zero?
The result is easy and "almost obvious":

$$V(r,t) = \frac{1}{4\pi\varepsilon_0} \int \frac{[\rho(r')]_{ret}}{|r-r'|} d\tau'$$

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi} \int \frac{[\vec{J}(r')]_{ret}}{|r-r'|} d\tau'$$

where $[\rho(r')]_{ret}$ means $\rho(r'; t' = t - \frac{|r-r'|}{c})$
i.e., ρ is evaluated at $t' = (t - \frac{|\vec{r}-\vec{r}'|}{c})$, for fixed $\vec{r}$; the integral goes over $\vec{r}'$.
Sim. for $\vec{A}$. This makes sense:
the V & $\vec{A}$ are launched at various t'
but all arrive at $\vec{r}$ at the same time,
and this time in the past depends on r' and r.

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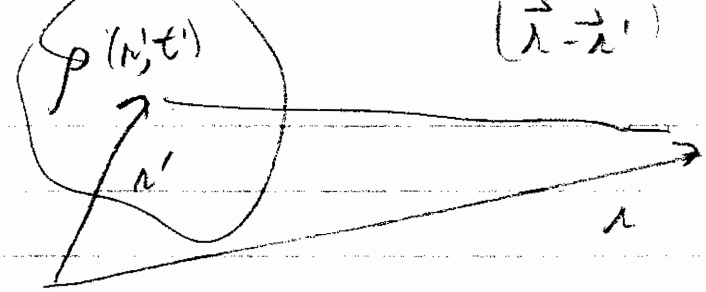


$\rho(r',t')$
$= \rho(r', t' = t - \frac{|\vec{r}-\vec{r}'|}{c})$

Why is this true?

① Put a small spherical ρ at $r=0$, (i.e., delta f'n. at origin), which produces a V satisfying

$$\nabla^2 V + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0 \quad \text{everywhere } \underline{\text{except}}$$

at $r=0$, or

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rV) + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0 \quad \text{or}$$

$$\frac{\partial^2}{\partial r^2}(rV) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rV) = 0 \quad \text{which}$$

is our old 1-d wave eq'n., for which

$rV = f(t - r/c)$ (or $t + r/c$!), and $f(x)$ is any function of x. So

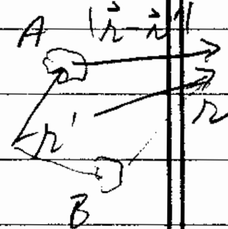
$V = f(t - r/c)/r$. Now we have the a{j}est f to solve wave eq'n <u>with source</u> at origin $\nabla^2 V + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\rho(r)/\varepsilon_0$, remembering $\rho(r) = 0$ for $r \neq 0$. What happens: $\nabla^2 V$ and $\rho(r)$ blow up at $r=0$, $\partial^2 V/\partial t^2$ is finite, so just have to solve $\nabla^2 V = -\rho/\varepsilon_0$ near $r=0$, which we know: $V = \int \rho(\vec{r})d\vec{r} / 4\pi\varepsilon_0 r$ (Poisson's eq. for point charge at origin) $= f(t)/r$, so $f(t) = \int \rho(t)\, d\vec{r}/4\pi\varepsilon_0$, and $V = \int \rho(t)\, d\vec{r}/4\pi\varepsilon_0 r$ near $r=0$. So now for arbitrary r and t,

$$V = \frac{\int \rho(t - r/c)\, d\vec{r}}{4\pi\varepsilon_0 r}, \text{ where } \rho \text{ is non-zero only near } r = 0.$$

(c) This is result for source at $r=0$. Well, let's put it at $r=r'$:

$$V = \frac{1}{4\pi\varepsilon_0}\int_{\text{near } A} \frac{\rho_A\left(t-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}\, d(\vec{r}-\vec{r}')$$

or $d\vec{r}'$, since r is fixed.



Now add a source at B

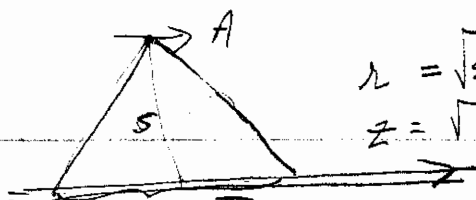
$$+\ \frac{1}{4\pi\varepsilon_0}\int_{\text{near } B} \frac{\rho_B\left(t-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}\, d\vec{r}' \ + \cdots$$

or all sources everywhere

$$V = \frac{1}{4\pi\varepsilon_0}\int_{\text{all } r'} \frac{d\vec{r}'\, \rho\left(t-\frac{|\vec{r}-\vec{r}'|}{c};\vec{r}'\right)}{|\vec{r}-\vec{r}'|}$$

QED.

Sim for $\vec{A}$.



$r = \sqrt{s^2+z^2}$ or
$z = \sqrt{r^2 - s^2} = \sqrt{ct^2 - s^2}$

Ex: Long wire I_0, $t > 0$ only,

See current from this section of wire only

(Exercise only!)
What are E & B?
$V = 0$, no charge

$$\vec{A} = \hat{z}\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} [J]_{ret}/r \; dr$$

$$= \hat{z}\frac{\mu_0}{4\pi}\int_{-\sqrt{ct^2-s^2}}^{+\sqrt{ct^2-s^2}} \frac{I(t - r/c)}{r}\, dz$$

at time t,
$t' = t - r/c$
So $t' = 0 \Rightarrow t = r/c$

anything sent from $r > ct$ will have left before current is turned on (before $t' = 0$).

$$= \hat{z}\frac{\mu_0}{4\pi} 2 \underbrace{\int_0^{\sqrt{(ct)^2-s^2}} \frac{I_0}{\sqrt{s^2+z^2}}\, dz}$$

$$= \hat{z}\frac{\mu_0 I_0}{2\pi} \ln\left(z + \sqrt{z^2+s^2}\right)\Big|_0^{\sqrt{(ct)^2-s^2}}$$

$$= \hat{z}\frac{\mu_0 I_0}{2\pi} \ln\left(\frac{ct + \sqrt{(ct)^2-s^2}}{s}\right)$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = \hat{z}\frac{\mu_0 I}{2\pi}\frac{s}{\sqrt{(ct)^2-s^2}}$$ (ugh!)

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\partial A_z}{\partial s}\hat{\phi} = \frac{\mu_0 I}{2\pi s}\frac{ct}{\sqrt{(ct)^2-s^2}}\hat{\phi}$$ (ugh!)

Both valid only for $ct > s$. For $ct < s$, both are zero. As $ct \to \infty$,

$$\vec{E} = 0 \qquad \vec{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\phi}$$