Problem set -5

16. **Phase shift and resonances**.

(a) Near an isolated resonance, the phase shift can be expressed as

$$d_\ell = a + \tan^{-1}\frac{\Gamma/2}{E - E_0}$$

Sketch the cross section vs E near the resonance for a=0, $p/4, p/2, p$. Express the energy in terms of the reduced parameter $e = \frac{E - E_0}{\Gamma/2}$. An isolated resonance can be parameterized in the form of "Fano profile" where the cross section takes the form

$$s = s_0 \frac{(e+q)^2}{1+e^2}$$

Find the values of q for the four cases indicated.

(b) Show that $\hbar(dd_\ell / dE)$ evaluated near the resonace is a measure of the lifetime. Discuss why it is not so easy to read the resonance width and resonance position from the cross section vs energy directly unless the parameter a in part (a) is near zero.

17. **Two simple mathematical exercises**. Do Exercises 13.19 and 13.20 of Merzbacher. (I did not do this part in the lecture. But you should be able to follow the book and work out the problems.)

18. In this simple exercise you will **practice the usage of rotational matrix or rotational operator for spin ½ particles.**

Let $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Clearly a is a "spin-up" state vector with respect to the z-direction.

(a) One can construct a spin-up state vector in the x-direction by a rotation operator. Use eq. (16.62) of Merzbacher to obtain a spin-up state vector in the x-direction. Show that your anwer is correct by proving that it is an eigenstate of the s_x operator with the correct eigenvalue.

(b) The polarization $\vec{P}$ of a state vector c is defined by $\vec{P} = < c \,|\, \vec{s} \,|\, c >$. Show that the polarization for the states in part (a) indeed are unit vectors pointing in the z-axis and x-axis, respectively.

(c) Calculate the polarization of the state $c = \cos\frac{q}{2} a + e^{if} \sin\frac{q}{2} b$. Show that the polarization vector of this state is a unit vector with polar angles (q, f).

(d) Do exercise 16.12 of Merzbacher showing that the vector formed from the three Pauli matrices indeed transform like a vector. This shows that spin is a vector operator in the spinor space.

19. **Addition of angular momenta**.

In the lecture we illustrate how to obtain Clebsch-Gordon coefficients by adding two spin ½ angular momenta.

(a) Try to add two spin 1 angular momenta using the same procedure. Check your answers against the results from Tables in some books (You need to find them yourself, but they are in some quantum mechanics books or in some monographs on angular momentum.). Or you can use the weblink I provided to get the 3j symbols and from there to get the CG coefficients.

(b) If the angular momentum is due to two identical particles, identify the symmetric and antisymmetric states under particle exchange.

(c) Now suppose that we are adding two orbital angular momenta, each with $\ell = 1$. One of the coupled states has total L=0. Show that the total orbital wave function for this state is proportional to $P_1(\cos\theta_{12})$ where θ_{12} is the angle between the two particles.

20. **Application of Wigner-Eckhart theorem** to transitions between Zeeman levels.

Suppose that the excited state of an atom has j=3/2 and its ground state has j=1/2. Let the energy separation between the two states be A. In a magnetic field each j will be splitted into (2j+1) components. Let the upper level shift be given by am and the lower level the shift is given by bm' where m and m' are the magnetic quantum numbers of the upper state and the lower state, respectively. Note that the Zeeman shift should be much less than the level separation A. (We will talk about how to calculate the splitting later.)

(a) The upper Zeeman levels now can decay to the lower Zeeman levels if m and m' differ by zero or one (the dipole selection rule). Calculate how many lines you are to observe?

(b) The transition rate is proportional to $|< jm \,|\, T_{1q} \,|\, j'm' >|^2$ (T is the dipole operator or tensor operator of rank 1, but its precise form is not important). Use Wigner-Eckhard theorem to calculate the relative intensities of the lines derived in part (a).