Problem set -3

8. **A simple shell model for nuclei and for atoms** (and later for molecules)

In this exercise we will assume that each proton or neutron in a nucleus is bound in a deep spherical potential well assuming of infinite depth. We will derive the simple shell model for the nucleus.

(a) For the infinite spherical potential well of width a, calculate the energy levels of the first 8 levels. Note that you need to consider for different angular momentum ℓ. The first level of $\ell = 0$ is 1s, second one is 2s, .. The first ℓ=1 is called 1p, .. and so on. Show that the order of the levels are 1s, 1p, 1d, 2s, 1f, 2p, 1g, 2d,.. and draw the energies in real scale. Identify the degeneracy of each level.

(b) Since proton and neutron are spin ½ particles, each level can accommondate 2 of each. Find the magic number of protons and neutrons.

(c) You can work out something similar for atoms. Starting with atomic hydrogen, all the levels within the same n are degenerate. If the potential is not a pure Coulomb potential, then the degeneracy in ℓ is removed. Indicate the relative energy levels in atoms. Note that in atoms the lowest p level is called 2p, the lowest d level is called 3d. It is just a convention. Find the magic numbers of atoms-- and relate this to the closed-shell atoms.

Note: Magic numbers are found in all kind of problems and most recently, in the study of simple clusters. Their origin can come from the geometry of packing "spherical" atoms as tightly as possible, or from the quantum origin of free electrons "swimming" in a spherical shell. Here is a link that explains some of these properties of the clusters.

http://tabish.freeshell.org/physics/Sr/

9. **The Block wave functions** .This comes from Merzbacher Exercise 13.7, page 295. .

If the potential has translational invariance $V(\vec{r}+\vec{R}) = V(\vec{r})$, where R is a constant vector, show that the solution of the integral form of the Schrordinger equation are Block wave functions satisfying the relation

$$\Psi_k(\vec{r}+\vec{R}) = e^{i\vec{k}\bullet\vec{R}}\Psi_k(\vec{r})$$

and that scattering amplitude vanishes unless $\vec{q} = \vec{k} - \vec{k}'$ is a reciprocal lattice vector satisfying the condition

$$\vec{q}\bullet\vec{R} = 2n\pi$$

which is the Laue condition.

A similar problem occurs in the time-dependent problem for a monochromatic laser acting on an atomic or molecular system. The time dependent Hamiltonian can be expressed as

$$H(x,t) = H_0(x) + V(x)e^{i\omega t}$$

Show that the time-dependent Schrodinger equation has the solution in the form

$$\psi(x,t) = e^{i\varepsilon t}\sum_n U_n(x)e^{in\omega t}$$

where ε is a complex energy, obtained from solving the eigenvalues of the coupled equations in $U_n(x)$. This is called **Floquet theory** for multiphoton ionization and the complex part of the eigenenergy gives the ionization rate and the real part is called the ac Stark shift energies. We will come back to this problem after we talk about time dependent Schrodinger theory.

10. ***More on scattering length***. You need to follow the notation from the lecture notes. In the class we have shown that the phase shift for s-wave scattering from an attractive square well can be written as

$$\tan\delta_0 = \frac{k\tan pa - p\tan ka}{p + k\tan ka\tan pa}$$

and that the scattering length $a_s = a - \frac{\tan\lambda a}{\lambda}$ and that the wavefunction near zero energy is used to understand the meaning of the scattering length.

(a) For simplicity, consider a=1 and vary the range of λ from 1.0 to 5.0 and plot the scattering length vs λ.

(b) plot the wavefunction u(r)=r R(r) for k=0.1 and several values of λ for positive and negative scattering lengths and confirm the geometric meaning of scattering length.

(c) Locate the bound states within the range of λ from 1.0 to 5.0.

(d) For small k, show the equation

$$k\cot\delta_0 = -\frac{1}{a_s} + \frac{1}{2}r_0k^2$$

This equation is called the **effective range theory**. This shows that at low energies the s-wave scattering is determined by the two parameter, a_s and the effective range r_0.